A CONNECTIVITY MAP $f: S^n \to S^{n-1}$ DOES NOT COMMUTE WITH THE ANTIPODAL MAP

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1. Introduction

In 1956 in [8] J. Nash asked whether Brouwer's fixed point theorem held for connectivity maps. In 1957 in [4] O. H. Hamilton answered this question affirmatively. Since then a number of papers on connectivity maps and non-continuous functions has appeared, notably those of J. Stallings [9] and G. T. Whyburn [12], [13], [14]. However, in [5] I pointed out that no further theorems of algebraic topology had been proved for connectivity maps. In this article we prove the proposition stated in the title, which for continuous functions is equivalent to the Borsuk-Ulam antipodal point theorem (see pp. 138–139 of [7]).

2. A disconnection theorem

In this section X will denote a Peano continuum.

We shall say that a subset L of X is *semi-closed in* X if for each convergent sequence $K_1, K_2, \cdots$ of components of L, $\lim K_i$ is a single point or a subset of L. This definition is given on p. 131 of [11]. It follows from the definition that the components of a semi-closed set L are closed in X. Further, we notice from (5.2), p. 132 of [11], that the components of a semi-closed set L and the single points of $X - L$ form an usc decomposition of X.

We shall say that a subset L of X *disconnects two points* p, q *in* X if p and q lie in different components of $X - L$. This definition is given on p. 439 of [10]. In [14] and [5] the same definition is given, but in these papers the phrase "weakly separates" is used instead of "disconnects."

Before proving the disconnection theorem of this section (theorem (2.1)), we state the following result.

LEMMA (2.1). *Let X be a unicoherent Peano continuum and T an involution on X. Let L be a subset of X with closed components such that $T(L) = L$ and L separates x, $T(x)$ in X for all $x \notin L$. Then there is a component K of L such that $T(K) = K$ and K separates x, $T(x)$ in X for all $x \notin K$.*

This is proved as a theorem in [6]. It is a generalization of the lemma of [2], where the set L is assumed to be compact.

THEOREM (2.1). *Let X be a unicoherent Peano continuum and T an involution on X. Let L be a semi-closed subset of X such that $T(L) = L$ and L disconnects x, $T(x)$ in X for all $x \notin L$. Then there is a component K of L such that $T(K) = K$ and K separates x, $T(x)$ in X for all $x \notin K$.*

PROOF. The collection of disjoint closed sets consisting of the components of L

and the single points of $X - L$ is an usc decomposition of X. Let $\pi: X \to \pi(X)$ be the monotone projection from X onto the decomposition space $\pi(X)$.

By (2.21), p. 138 of [11], $\pi(X)$ is a unicoherent Peano continuum. Further, it follows from (2.2), p. 138 of [11], that $\pi(L)$ is totally disconnected. Thus, by lemma 1 of [1], the quasi-components of $\pi(X) - \pi(L)$ are connected. However, by (5.3), p. 132 of [11], $\pi \mid X - L : X - L \to \pi(X - L)$ is a homeomorphism. Thus the quasi-components of $X - L$ are connected.

But this means that L separates x, $T(x)$ in X for all $x \notin L$. Since L has closed components, it now follows from lemma (2.1) that there is a component K of L such that $T(K) = K$ and K separates x, $T(x)$ in X for all $x \notin K$.

3. The main theorem

In this section we denote by S^n the set of all points $x = (x_1, x_2, \cdots, x_{n+1})$ such that $\sum_{i=1}^{n+1} x_i^2 = 1$ in Euclidean $(n + 1)$-space E^{n+1}. We identify the set of all points $x = (x_1, x_2, \cdots, x_{n+1})$ in E^{n+1} defined by $x_{n+1} = 0$ with E^n. Finally we denote by T the antipodal map on S^n defined by $T(x) = -x$.

We need the following lemma in order to prove theorem (3.1), on which the main theorem is based.

LEMMA (3.1). *If $F_1, F_2, \cdots, F_n$ are self-antipodal closed sets in S^n and each F_i separates x, $-x$ in S^n for all $x \notin F_i$, then $\bigcap_{i=1}^n F_i \neq \emptyset$.*

PROOF. Suppose that $\bigcap_{i=1}^n F_i = \emptyset$, and write $S^n - F_i = G_i \cup TG_i$, where G_i and TG_i are disjoint open sets. Then $G_1, TG_1, G_2, TG_2, \cdots, G_n, TG_n$ is a finite open covering of S^n. Thus there is a finite closed covering $H_1, TH_1, H_2, TH_2, \cdots, H_n, TH_n$ of S^n such that $H_i \subset G_i$ for each i. However, $H_i \cap TH_i = \emptyset$ for each i, and this contradicts theorem (21.3), p. 138 of [7].

THEOREM (3.1). *If $L_1, L_2, \cdots, L_n$ are self-antipodal semi-closed sets in S^n and each L_i disconnects x, $-x$ in S^n for all $x \notin L_i$, then $\bigcap_{i=1}^n L_i \neq \emptyset$.*

Proof. We suppose that $n > 1$. Then for each i there is by theorem (2.1) a self-antipodal component K_i of L_i such that K_i separates x, $-x$ in S^n for all $x \notin K_i$. By lemma (3.1), $\bigcap_{i=1}^n K_i \neq \emptyset$. Thus $\bigcap_{i=1}^n L_i \neq \emptyset$, which proves the theorem.

A function $f: X \to Y$ is called a *connectivity map* if for each connected set C in X the graph of the restricted function $f \mid C: C \to Y$ is a connected subset of $X \times Y$.

Notice that a connectivity function preserves connectedness. Notice also that the proof of theorem (3.1) of [3] establishes the following proposition: if $f: X \to Y$ is a connectivity function, where X and Y are Peano continua, and if F is a closed subset of Y, then $f^{-1}(F)$ is a semi-closed subset of X.

THEOREM (3.2). *There is no connectivity map $f: S^n \to S^{n-1}$ which commutes with the antipodal map.*

Proof. The case $n = 1$ is obvious, because a connectivity map preserves connectedness. Thus, taking $n > 1$, we suppose that there is a connectivity map $f: S^n \to S^{n-1}$ which commutes with the antipodal map, so that $f(-x) = -f(x)$ for all $x \in S^n$.

Let A_i, for each $i \leq n$, be the set of all points $x = (x_1, x_2, \cdots, x_n)$ in S^{n-1} such that $x_i = 0$. Then A_i is a self-antipodal set which separates x, $-x$ in S^{n-1} for all $x \in S^{n-1} - A_i$. Further, $\bigcap_{i=1}^{n} A_i = \varnothing$.

Now put $L_i = f^{-1}(A_i)$. Then L_i is a self-antipodal semi-closed subset of S^n. In addition, if $x \in S^n - L_i$ then L_i disconnects x, $-x$ in S^n, because f preserves connectedness and A_i separates $f(x), f(-x) (= -f(x))$ in S^{n-1}. Thus, by theorem (3.1), $\bigcap_{i=1}^{n} L_i \neq \varnothing$. But this is impossible because $\bigcap_{i=1}^{n} A_i = \varnothing$, and so the theorem is proved.

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