ON σ-CONNECTED SETS

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1. Introduction

The purpose of this paper is to find the best possible generalization of the following theorem, due to Sierpiński:

1.1 *No Hausdorff continuum can be expressed as a countable union of closed, non-empty, mutually disjoint subsets.* (See [2] 2.49 or [4], p. 173, Th. 6).

At the end we give a series of examples to exhibit possible limitations to this task.

In order to get two immediate generalizations, we need some definitions.

1.2 Let X be an arbitrary topological space. A sequence $C_1, C_2, \cdots$ of subsets of X is a *σ-partition* of X if the C_i's are non-empty and mutually disjoint and their union is X. A σ-partition of X is *closed* (resp., *compact*; resp., *connected*) if each element is closed (resp., compact; resp., connected). A connected space X is *σ-connected* if it has no closed σ-partition. If $p \in X$, the *constituant* of p is the union of all compact connected subsets of X containing p and the *σ-component* of p is the union of all σ-connected subsets of X containing p.

The following three properties of σ-connected sets can be proved easily. We omit their proof.

1.3 *If $\{C_\alpha \mid \alpha \in I\}$ is a family of σ-connected subsets of a space X with non-empty intersection, then $\bigcup \{C_\alpha \mid \alpha \in I\}$ is σ-connected.*

1.4 *If $A \in 2^X$ is σ-connected and $A \subset A_1 \subset A^-$, then A_1 is σ-connected. In particular, the closure of a σ-connected set is σ-connected.*

1.5 *If $H_1, H_2, \cdots$ is a closed σ-partition of a space X and A is a σ-connected subset of X, then $A \subset H_n$ for some n.*

It is clear that two different constituants of a space X must be disjoint. By 1.3 and 1.4, each σ-component of X must be closed and σ-connected. Besides, two different σ-components of X must be disjoint. Finally, by 1.1 and 1.3, each constituant of a Hausdorff space is σ-connected.

Theorem 1.6. *No locally compact, connected, Hausdorff space X has a compact σ-partition.*

Proof. If $C_1, C_2, \cdots$ were such a partition of X, then $\{\infty\}, C_1, C_2, \cdots$ would be a closed σ-partition of the one point compactification $X^+ = X \cup \{\infty\}$ of X, and this would contradict 1.1.

Theorem 1.7. *Every connected space which is a finite union of σ-connected subspaces is σ-connected.*

Proof. If $X = D_1 \cup D_2 \cup \cdots \cup D_r$, where each D_i is σ-connected, then $X = C_1 \cup C_2 \cup \cdots \cup C_r$, where C_i is the σ-component of X containing D_i. Since X is connected, we must have $C_1 = C_2 = \cdots = C_r = X$, so that X is σ-connected.

This theorem has two important corollaries:

1.8 *Every connected Hausdorff space X with only a finite number of constituants is σ-connected.*

1.9 *Every locally compact, locally connected, connected, Hausdorff space X is σ-connected.*

Proof. For every pair of points $a, b \in X$, there exists a chain $R_1, R_2, \cdots, R_n$ from a to b consisting of regions with compact closure. Then $R_1^- \cup R_2^- \cup \cdots \cup R_n^-$ is a continuum in X containing $\{a, b\}$. This implies that X has only one constituant.

As examples 4.1, 4.2 and 4.3 show, Corollary 1.9 is false if we omit any of the assumptions.

Some other easily proved properties of σ-connected sets are described in the next

THEOREM 1.10
- a) *Every continuous image of a σ-connected space is σ-connected.*
- b) *Every product of σ-connected spaces is σ-connected.*
- c) *A connected space X is σ-connected if and only if every point of X has a σ-connected neighborhood.*

S. Mazurkiewicz proves in [5] the following results:

1) *There exists a closed connected subset of the plane which is not σ-connected.*
2) *No closed connected subset of the plane admits a closed connected σ-partition.*

It is well known that the last result above is false in R^3. In fact, in 4.3 we provide an example of a closed connected subset of R^3 which admits a closed connected σ-partition.

2. Main Theorem

For each point p in an arbitrary topological space X, we define, by transfinite induction.*

$C^0(p)$ = constituant of X containing p, and for each ordinal number $\alpha > 0$,
$C^\alpha(p) = \{x \in X \mid$ there exists a finite chain $K_1, K_2, \cdots, K_m$ from p to x such that $K_i = C^{\alpha_i}(p_i)^-$, where $\alpha_i < \alpha, p_i \in X$ for $i = 1, 2, \cdots, m\}$.

In a Hausdorff space each $C^\alpha(p)$ is σ-connected. In any space, $C^\alpha(p)^- \subset C^\beta(p)$ whenever $\alpha < \beta$.

LEMMA 2.1. *Let X be a topological space and let Γ be the least infinite cardinal*

* For a complete account on the algebra of cardinals and ordinals, see [1], Chap. 2.

number such that X has a basis of cardinality $<\Gamma$. If $p \in X$, there exists an ordinal $\alpha < \Gamma$ such that $C^{\alpha}(p) = C^{\beta}(p)$ for each ordinal β such that $\alpha \leq \beta < \Gamma$.

Proof. Assume the theorem is false. Then, for each $\alpha \in \Gamma$, there exists a unique $f(\alpha) \in \Gamma$ such that $f(\alpha) > \alpha$ and $C^{\alpha}(p) \neq C^{f(\alpha)}(p)$, but $C^{\alpha}(p) = C^{\beta}(p)$ for each $\beta \in [\alpha, f(\alpha))$. The function $g:\Gamma \to \Gamma$ defined as $g(\alpha) = f(f(\alpha))$ has the following property:

If $\alpha \in \Gamma$, then $C^{\alpha}(p)^{-} \neq C^{g(\alpha)}(p)$, for

$$C^{\alpha}(p)^{-} \subset C^{f(\alpha)}(p) \neq C^{f(f(\alpha))}(p) = C^{g(\alpha)}(p).$$

The family $\mathcal{G} = \{H \mid \text{for some } \alpha \in \Gamma,\ H = C^{g(\alpha)}(p)^{-}\}$ has cardinality Γ, for otherwise there would exist $\beta \in \Gamma$ such that $C^{\beta}(p)^{-} = C^{g(\alpha)}(p)^{-}$ for each α in a subset T of Γ of cardinality Γ and, therefore, taking $\alpha \in T$, where $\alpha > g(\beta)$ we would obtain

$$C^{\beta}(p)^{-} \neq C^{g(\beta)}(p) \subset C^{\alpha}(p) \neq C^{g(\alpha)}(p),$$

contradicting the equality $C^{\beta}(p)^{-} = C^{g(\alpha)}(p)^{-}$.

$\mathcal{G}$ is then a family of closed sets in X, linearly ordered by inclusion, of cardinality Γ. But this cannot happen in a space having a basis of cardinality $<\Gamma$.

Let $p \in X$ and Γ be as in 2.1. We define $D(p) = C^{\alpha}(p)$, where α is the first ordinal in Γ such that $C^{\alpha}(p) = C^{\beta}(p)$ for each $\beta \in [\alpha, \Gamma)$.

Observe that each $D(p)$ is closed in X, for if $D(p) = C^{\alpha}(p)$, then $C^{\alpha}(p)^{-} \subset C^{\alpha+1}(p) = C^{\alpha}(p)$. Also, if $D(p) \cap D(q) \neq \Phi$, necessarily $D(p) = D(q)$. For if $D(p) = C^{\alpha}(p)$, $D(q) = C^{\beta}(q)$ and, say, $\alpha \leq \beta$, then

$$C^{\alpha}(p)^{-} = C^{\alpha}(p) \subset C^{\beta+1}(q) \subset C^{\beta+2}(p) = C^{\alpha}(p),$$

so that $C^{\alpha}(p) = C^{\beta+1}(q) = C^{\beta}(q)$.

THEOREM 2.2. 1) *Let X be a connected Hausdorff space. If there exists a finite set of points $p_1, p_2, \cdots, p_n$ in X and an ordinal number α such that*

$$X = C^{\alpha}(p_1) \cup \cdots \cup C^{\alpha}(p_n),$$

then X is σ-connected.

2) *Let X be a σ-connected Hausdorff space. If there exists a sequence $p_1, p_2, \cdots$ in X and an ordinal number α such that $X = \bigcup_{i=1}^{\infty} C^{\alpha}(p_i)$, then, for some $p \in X$, $X = D(p)$.*

Proof. 1) is a direct consequence of 1.7, 1.3 and 1.4. As for 2), notice that $X = D(p_1) \cup D(p_2) \cup \cdots$. Since X is σ-connected and the sets $D(p_i)$ are closed and disjoint if different, we have $X = D(p_i)$ for each index i.

3. Enlarging σ-connected sets

In this section we prove that a connected Hausdorff space is σ-connected if it contains a σ-connected subset with a certain property. (See corollaries 3.5 and 3.6 below). This is another direction on which an improvement of 1.1 can be achieved.

We shall make use of the following theorem on separation of T_2-spaces:

3.1 *Let X be a locally compact Hausdorff space with compact components and let A, B be subsets of X such that A is compact and B is closed. If no component of X intersects both A and B, then there exist two disjoint subsets C, D of X such that C is compact, D is closed, $A \subset C$, $B \subset D$ and $X = C \cup D$.*

Now we are able to prove:

THEOREM 3.2. *Let $K_1, K_2, \cdots$ be a closed σ-partition of a connected T_2-space X. If L is a locally compact subset of X with compact components, then*

$$K_n \sim \text{Int}\, L \neq \Phi$$

for each n.

Proof. If Int $L = \Phi$, there is nothing to prove. Our hypothesis, combined with 1.1, implies that $L \neq X$. Hence we can assume that $\Phi \neq \text{Int}\, L \neq X$. Proceeding by contradiction, assume that Int L contains some K_n, say, $K_1 \subset \text{Int}\, L$. Let M be a component of L intersecting K_1. Necessarily, $M \cap \text{Fr}\,(\text{Int}\, L) \neq \Phi$, for if $M \subset \text{Int}\, L$, 3.1 guarantees the existence of a separation $L = C \cup D$, where C is compact, $M \subset C \subset \text{Int}\, L$ and $L \cap \text{Fr}\,(\text{Int}\, L) \subset D$. Then

$$X = C \cup [D \cup (X \sim L)]$$

would be a separation of X, contradicting its connectedness. Then we have $M \cap \text{Fr}\,(\text{Int}\, L) \neq \Phi$. But 1.5 implies that $M \subset K_1 \subset \text{Int}\, L$, a contradiction.

COROLLARY 3.3. *If G is an open subset of X with compact closure, then $K_n \sim G \neq \Phi$ for each n.*

Proof. Take $L = G^-$ in the theorem.

COROLLARY 3.4. *If besides being connected and Hausdorff, X is locally compact, then no K_n can be compact.*

Proof. A compact K_n would have a compact neighborhood, contradicting previous corollary.

COROLLARY 3.5. *Let X be a connected T_2-space. Then X is σ-connected if and only if X has a σ-connected subspace S such that $L = X \sim S$ is locally compact and has compact components.*

COROLLARY 3.6. *Let X be a locally compact, connected T_2-space. Then X is σ-connected if and only if X has a σ-connected region whose complement has compact components.*

4. Examples

4.1 *There exists a connected, locally connected Hausdorff space X which is not σ-connected.*

Proof. Let X be the set of natural numbers. Denote by $\mathcal{S}$ the set of all arithmetic progressions $\{a + n\,d \mid n = 0, 1, \cdots\}$ where $a \in X$ and d is a prime greater than a. If τ is the topology of X having $\mathcal{S}$ as a subbasis, then (X, τ) is connected, locally connected, Hausdorff and countable, so this space cannot be σ-connected. (See [3]).

4.2 *There exists a T_1-locally connected continuum which is not σ-connected.*

Proof. Just take the integers with the smallest T_1-topology.

4.3 *There exists a locally compact, connected T_2-space which is not σ-connected. More than that, there exists a closed connected subset of R^3 which admits a closed connected σ-partition.*

Proof. For each positive integer n, let $K_n \subset R^3$ consist of the lines $x = \pm 2n$ on the plane $z = 0$; the lines $x = \pm 2, x = \pm 4, \cdots, x = \pm 2n$ on the plane $z = 1/n$; the segment $\{(x, n, 1/n) \mid -2n \leq x \leq 2n\}$ and the two circles described on the segments $\{(\pm 2n, n, z) \mid 0 \leq z \leq 1/n\}$ as diameters and lying in the plane $y = n$. The sets K_n are closed, connected and mutually disjoint. Their union X is also closed (in R^3) and connected. Therefore X provides our example.

The following example was given to me by Dr. Edwin E. Moïse:

4.4 *There exists a connected, punctiform subset of R^2 which is not σ-connected.*

Proof. Let $\alpha_1, \alpha_2, \cdots$ be a family of closed arcs in the unit circle S^1 such that:
i) The arcs α_j are mutually disjoint.
ii) Their union is a dense subset of S^1.

The arcs α_j can be obtained by induction as follows: let $p_1, p_2, \cdots$ be a countable dense subset of S^1. Let α_1 be a closed arc in S^1 containing p_1. Assuming that $\alpha_1, \alpha_2, \cdots, \alpha_n$ have been constructed in such a way that $\alpha_i \cap \alpha_j = \Phi$ for $i \neq j$ and with $\{p_1, p_2, \cdots, p_n\} \subset \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n$, let k_n be the smallest index such that

$$p_{k_n} \notin \bigcup_{j=1}^{n} \alpha_j .$$

Define α_{n+1} as any closed arc in $S^1 \sim \bigcup_{j=1}^{n} \alpha_j$ containing p_{k_n}. Let H_n be the set of points in the closed unit disk comprised between the circles $x^2 + y^2 = 1/n$, $x^2 + y^2 = 1$ and the segments joining the origin to the end points of α_n. Let D be Kuratowski's punctiform set (See [4], p. 135). Clearly D can be mapped homeomorphically into H_n in such a way that the image contains a dense subset of $\operatorname{Fr} H_n$. If D_n is the image set, $X = \{(0, 0)\} \cup D_1 \cup D_2 \cup \cdots$ provides the example.

Note. Keeping the notation of 4.4, we can see that 1.6 fails if we drop the assumption of local compactness. Indeed, $H_1, H_2, \cdots, \{(0, 0)\}$ constitutes a compact, connected σ-partition of the connected set $Y = \{(0, 0)\} \cup H_1 \cup H_2 \cup \cdots$.

4.5 *Let X be an indecomposable metric continuum and let K be a proper subcontinuum of X. Then $X \sim K$ has uncountably many constituants and each of them is dense in X. Therefore, by 1.4, $X \sim K$ is σ-connected.*

Proof. Let M be the composant of X containing K. Then every composant of X different from M is a constituant of $X \sim K$. But X has uncountably many composants and each of them is dense in X. (For a proof of these results see [4], p. 204–214).

We finish this paper with two conjectures:

Let X be a connected, locally compact T_2-space. Then X is σ-connected if and only if for some $p \in X$, we have $X = D(p)$. (Compare with 2.2.)

Every connected and locally connected subset of the n-dimensional Euclidean space R^n is σ-connected.

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