AN ARC AND A CUBE

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The following is an example of an arc in $L_2[0, 1]$ having the property that its closed convex hull is a Hilbert cube and also that its nonzero points are linearly independent.

Let F be the function from $[0, 1]$ into $L_2[0, 1]$ defined by $F(t) = f_t$ where $f_0(x) = 0$ if $0 \leq x \leq 1$, $f_1(x) = 1$ if $0 \leq x \leq 1$ and if $0 < t < 1$ then $f_t(x) = 1$ if $0 \leq x \leq t$ and $f_t(x) = 0$ if $t < x \leq 1$.

That F is a homeomorphism follows easily from the fact that $\| F(a) - F(b) \|^2 = | a - b |$. Hence we have an arc $\alpha = F([0, 1])$.

Since each non-zero point in the arc α is a simple step function of height one, it follows that the non-zero points in α form a linearly independent set.

Let A denote the convex hull of α and let B denote the set of all non-negative, non-increasing step functions on $[0, 1]$ which are continuous from the left and which are bounded above by one. It is clear that every constant function in B is in A. Suppose now that f is in B and the range of f has exactly two values a and b with $1 \geq a > b \geq 0$. Let $x_0 = 0 < x_1 < x_2 = 1$ denote the endpoints of the segments on which f assumes the distinct values. Define a function g on $[0, 1]$ which is in A, by $g(x) = (a - b)/(1 - b)$ on $[x_0, x_1]$ and $g(x) = 0$ on $[x_1, x_2]$. Let h be the function defined on $[0, 1]$, which is identically one. Then $(1 - b)g + bh = f$ on $[0, 1]$ and hence f is in A. Suppose now that n is a positive integer and that every function in B having at most n values, is in A. Let f be function in B which has exactly $n + 1$ values. If the function f assumes the value 0 on the last step, then define a function g on $[0, 1/x_n]$ by $g(x) = f(x)(x_n)$, where $0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1$ denotes the endpoints of the segments on which f assumes the $n + 1$ distinct values. If we restrict g to $[0, 1]$ then the range of g has exactly n values and hence g restricted to $[0, 1]$ is in $A \cap B$. Therefore there are non-negative numbers $a_1, a_2, \cdots, a_k$ and non-negative numbers $y_1, y_2, \cdots, y_k$ where $0 \leq y_1 < y_2 < \cdots < y_k \leq 1$ such that $a_1 + a_2 + \cdots + a_k = 1$ and $g = a_1F(y_1) + \cdots + a_kF(y_k)$. It then follows that $f = a_1F(x_ny) + \cdots + a_kF(x_ny_k)$ *on* $[0, 1]$ and hence f is in A.

Suppose now that f is not 0 on $[0, 1]$. Let the smallest two numbers in the range of f be denoted by a and b with $b > a > 0$. Define a function g in A by $g = f$ on the first $n - 1$ steps and $g(x) = a$ on the last two steps. Define a function h in A by $h = f$ on the first $n - 1$ steps and $(b - a)/(1 - a/f(0))$ on the n^{th} step and 0 on the last step. Then $\alpha g + (1 - \alpha)h = f$ where $\alpha = a/f(0)$ and hence f is in $A \cap B$, which then concludes this portion of the argument and hence $A = B$. It follows then that the closed convex hull of α is compact.

Hence we have an arc α such that the closed convex hull of α is both compact and infinite dimensional from which it follows by corollary 1.3 in [3] that it is a Hilbert cube.

Suppose now that M is a nondense subset of α. Then $H = F^{-1}(M)$ is a nondense subset of [0, 1]. Hence there is an interval $[a, b] \subseteq (0, 1)$ such that $[a, b]$ does not intersect the closure of H. Let $a < c < d < b$ and $g = F(d) - F(c)$. Suppose h is in the linear extension of $F([0, a]) \cup F([b, 1])$. Note that h is constant on $[a, b]$. Let $h((a + b)/2) = z$.
Then

$$\begin{aligned}\| g - h \|^2 &= \textstyle\int_0^1 (g - h)^2 \\ &= \textstyle\int_0^a (g - h)^2 + \int_a^c (g - h)^2 + \int_c^d (g - h)^2 + \int_d^b (g - h)^1 \\ &\quad + \textstyle\int_b^1 (g - h)^2 \\ &= \textstyle\int_0^a h^2 + \int_a^c h^2 + \int_c^d (g - h)^2 + \int_d^b h^2 + \int_b^1 h^2 \\ &= \textstyle\int_0^a h^2 + z^2(c - a) + (1 - z)^2(d - c) + z^2(b - d) + \int_b^1 h^2 \\ &\geq z^2(c - a + b - d) + (1 - z)^2(d - c)\end{aligned}$$

Let $K_1 = c - a + b - d$ and $K_2 = d - c$. Then

$$\begin{aligned}\| g - h \|^2 &\geq [K_1/(K_1 + K_2)]^2 K_1 + [1 - K_1/(K_1 + K_2)]^2 K_2 \\ &= K_1^3 + K_2^3 + 2K_1K_2 \geq K_1^3 + K_2^3\end{aligned}$$

and hence g is not in the closure of the linear extensions of $F([0, a]) \cup F([b, 1])$.

We now have that the arc α has the property that if M is a non dense subset of α then the linear extension of M is not dense in $L_2[0, 1]$. The arcs described in both [4] and [5] have the property that the linear extension of every infinite subset is dense in the space.

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