HOMOLOGY OF KNOT GROUPS: I GROUPS WITH DEFICIENCY ONE[1]

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Abstract. The homology of a knot group with deficiency one is trivial above dimension 2.

§0. Introduction

A knot is an embedding $k: S^n \to S^{n+2}$ $(n \in N)$. The normal bundle $\nu(k(S^n) \subset S^{n+2})$ is always trivial; let $X = S^{n+2} - \text{Int } T$ where T is a tubular neighborhood of $k(S^n)$ (diffeomorphic to $S^n \times D^2$). Notice $\partial N \cong S^n \times S^1$. X is called the complement of k.

A *n-knot group* is a group of the form $\pi_1(X)$. The inclusion $\partial X \subset X$ induces a map of fundamental groups; the loop $(*) \times S^1$, where $* \in S^n$, is called a *meridian* and it is determined up to conjugation.

The purpose of this note is to describe the homology (with integral coefficients) of knot groups; by simple Alexander duality we know

(i) $H_1(\Pi) = Z$,
(ii) $H_2(\Pi) = 0$.

In fact from [5, 7], if $\tilde{\Pi}$ is the commutator subgroup of Π, we have

Proposition 1. *If* Π *is a* 1-*knot group and* $q \geq 2$

$$H_q(\Pi) = H_q(\tilde{\Pi}) = 0.$$

We extend that result to any n-knot group $(n \geq 2)$ of deficiency one (cf. [2; §7] and [7]).

§1. Higher dimensional knot groups

In [3] it is proved that any finitely presented group Π satisfying (i) and (ii) of §0 and the extra-condition

(iii) There exists an element of $\alpha \in \Pi$ whose normal closure is all of Π,

is an n-knot group for $n \geq 3$. In fact α turns out to be a meridian of Π. Condition (iii) is abbreviated by saying that Π has weight 1.

Little is known about 2-knot groups but if we change (ii) by the stronger condition

(ii′) Π has deficiency one, that is, it has a presentation with $r + 1$ generators and relations, for some $r \in N$,

then Π is a 2-knot group cf. th. 1, below).

Certainly, these are not all the 2-knot groups; in [1; ex. 12], it is proved that

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$Z_3 \times_\varphi Z$ is a 2-knot group, where the notation indicates the semidirect product of Z_3 with Z with automorphism $\varphi: Z_3 \to Z_3$ given by multiplication by 2. As we shall see, this group cannot satisfy (ii′).

§2. Adding handles

Suppose Π is a finitely presented group satisfying (i), (ii′) and (iii), then [4; p. 141] Π has a preabelian presentation of the following form

$$\langle \alpha, \beta_1, \cdots, \beta_r : \beta_j = B_j(\alpha, \beta_\tau) \rangle$$

where $1 \le j \le r$ and if Φ is the free group in the letters $\alpha, \beta_1, \cdots, \beta_r$, the words B_j lie in the commutator subgroup $\tilde{\Phi}$ by (iii) the group generated by the β_j and with relations $\beta_j = B_j(1, \beta_\tau)$ is trivial; this means that the map $\beta_j \to \beta_j(B_j(1, \beta_\tau))^{-1}$, $\alpha \to \alpha$, is an automorphism of Φ by [4; th. 3.3]. Taking the inverse automorphism, we get a new presentation for Π

$$(1) \qquad \langle \alpha, \beta_1, \cdots, \beta_r : \beta_j = B_j(\alpha, \beta_\tau) \rangle$$

where, again, $j = 1, \cdots, r$, $B_j \in \tilde{\Phi}$ and $B_j(1, \beta_\tau) = 1$.

THEOREM 1. *Let Π be a finitely presented group satisfying (i), (ii′) and (iii). Then, there exists an embedding $f: S^2 \to S^4$ such that, if X is the complement of f, $\pi_1(X) = \Pi$.*

Proof. We start with a particular example originally found by Sumners in [6]; consider the knot group

$$(2) \qquad \langle \alpha, \beta : \beta^2 \alpha \beta^{-1} \alpha^{-1} = 1 \rangle.$$

This presentation has the desired form (1). As in [6] take the standard embedding $D^3 \subset D^5$ with boundary the standard embedding $S^2 \subset S^4$. Attach $K = D^5 - D^3$ a trivial 1-handle $h^1 \approx D^1 \times D^4$. The boundary of $K \cup h^1$ has fundamental group

$$\langle \alpha, \beta : \rangle,$$

where α represents the meridian around D^3 and β the loop around h^1. In $\partial(K \cup h^1)$ represent the word $\beta^2\alpha\beta^{-1}\alpha^{-1}$ by an embedded loop γ as in the figure. If we put D^3 back in $K \cup h^1$, γ homotops, and thus, by [8], isotops to the loop β. This means that if we attach a 2-handle h^2 along a tubular neighborhood of γ, we recover an embedding

$$f: D^3 \to D^5 \cong K \cup h^1 \cup_\gamma h^2$$

and $f \mid \partial D^3 : S^2 \to S^4$ is a knot with group presented by (2).

In the general case, to K attach r 1-handles h_j^1 and in $\partial(K \cup \bigcup h_j^1)$ represent the words $\beta_j B_j^{-1}$ by loops γ_j. Attach 2-handles h_j^2 along neighborhoods of γ_j. If we put D^3 back in $K \cup \bigcup h_j^1$, the γ_j isotops to β_j because $B_j(1, \beta_\tau) = 1$, thus $\partial(D^5 \cup \bigcup h_j^1 \cup \bigcup h_j^2)$ is the standard sphere and $\partial(K \cup \bigcup h_j^1 \cup \bigcup h_j^2)$ is the complement of a knot with group presented by (1).

§3. Homology of Groups

We want to extend proposition 1 to all knot groups with deficiency one:

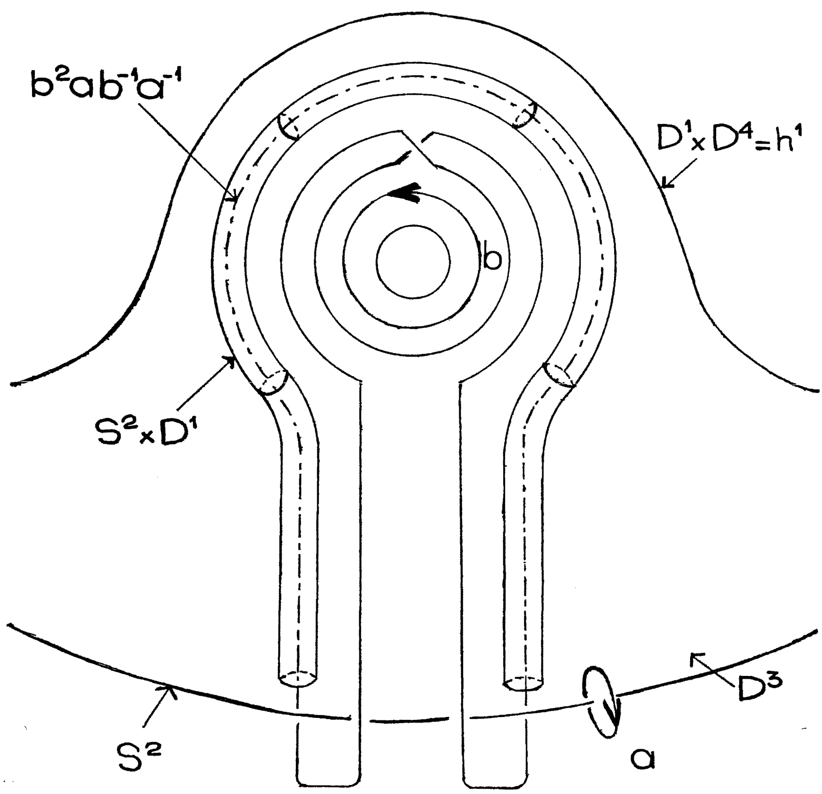
THEOREM 2. *Let* Π *be a finitely presented group satisfying* (*i*), (*ii*′) *and* (*iii*). *If* $\tilde{\Pi}$ *is its commutator subgroup, for* $q \geq 2$

$$H_q(\Pi) = H_q(\tilde{\Pi}) = 0.$$

Proof. Again we begin with Sumners' example: the trivial knot $S^2 \subset S^4$ can be extended to $\Delta^3 \subset S^4$ where Δ is a disk and $\partial\Delta = S^2$. With the notation of th. 1, $\Delta \subset \partial(K \cup h^1)$ but the arc γ pierces Δ in two points with opposite intersection numbers (cf. figure).

Remove two small disks, neighborhoods of $\Delta \cap \gamma$ and attach a tube $S^2 \times D^1$ to the punched disk to obtain $F \approx (S^1 \times S^1)_0$ where M_0 is obtained from the closed manifold M by punching a hole to it. $F \subset \partial(K \cup h^1 \cup h^2)$ and thus it is a Seifert manifold (cf. [2]) of the knot constructed in [6].

At this point we recall the notation introduced in [2; §0]: let Y be obtained from S^4 by cutting along F, that is by removing the interior of a regular neighborhood of F in S^4. Y is a compact manifold with boundary $F_0 \cup F_1$ where $F_t \approx F$



and $F_0 \cap F_1 = S^2$. The map $\nu_t: F \approx F_t \subset \partial Y \subset Y$ induces maps $\nu_t: \pi_1(F) \to \pi_1(Y)$. If both ν_t are monomorphic, F is called minimal.

In the present case it is not difficult to prove that $\pi_1(F) = \pi_1(Y) = Z$ and,

$$\nu_0(n) = n \quad \text{and} \quad \nu_1(n) = 2n \ (n \in Z).$$

By [5; th. 4.5.1], if Π is presented by (2), $\tilde{\Pi}$ is the infinite free product with amalgamations

$$\tilde{\Pi} = \cdots *_Z Z *_Z Z *_Z \cdots \tag{3}$$

where the ν_t are the amalgamating maps.

Let Λ be the integral group ring of Z; it can be identified to $Z[t, t^{-1}]$ where t is a variable. As in [2; §2] from [5; th. 4.5.1] we find an exact Mayer-Vietoris sequence

$$\to H_q(\pi_1(F)) \otimes \Lambda \xrightarrow{d} H_q(\pi_1(Y)) \otimes \Lambda \to H_q(\tilde{\Pi}) \to H_{q-1}(\pi_1(F)) \otimes \Lambda \to \cdots \tag{4}$$

where $d(\alpha \otimes 1) = (\nu_0(\alpha) \otimes t) - (\nu_1(\alpha) \otimes 1)$, whenever the maps $\nu_t: \pi_1(F) \to \pi_1(Y)$ are monomorphisms. Clearly in the particular case (2), the amalgamations in (3) are monomorphic and from (4) we conclude $H_q(\tilde{\Pi}) = 0$ for $q \geq 2$.

From the Wang sequence

$$\to H_q(\tilde{\Pi}) \xrightarrow{t-1} H_q(\tilde{\Pi}) \to H_q(\Pi) \to H_{q-1}(\tilde{\Pi}) \to \tag{5}$$

we conclude $H_q(\Pi) = 0$ in the same range.

For the general case, we again rely on the construction of th. 1. The disk $\Delta \subset S^4$ can be embedded in $\partial(K \cup \bigcup h_j^1)$ and the arcs γ_j pierce it with zero intersection number.

We know

$\pi_1(\partial(K \cup \bigcup h_j^1) = \langle \alpha, \beta_1, \cdots, \beta_r : \rangle$; write $\gamma_j = b_j \alpha^{\eta_1} \omega_1 \alpha^{\eta_2} \omega_2 \cdots \alpha^{\eta_s} \omega_s \alpha^{\eta_{s+1}}$

where $\Sigma \eta_i = 0$ and the ω_i are words in $\beta_1, \cdots, \beta_r$ only. Now we study $\gamma_j \cap \Delta$, a set of $\Sigma_i \mid \eta_i \mid$ points corresponding to the α^{η_i}. Thus, we can index the points by $x_1, \cdots, x_{|\eta_i|}$ $(i = 1, \cdots, s+1)$ by the way they occur in γ_j. Suppose $\eta_1, \cdots, \eta_i > 0$ and $\eta_{i+1} < 0$; we can add a tube T_1 to $\Delta - (N_1^{i+1} \cup N^i_{|\eta_i|})$, where N_j^i is a small (open) disk in Δ around x_j^i, joining ∂N_1^{i+1} to $\partial N^i_{|\eta_i|}$ along the loop representing ω_i. This reduces the number of intersection points by two. Now add a tube T_2 joining $x^i_{|\eta_i|-1}$ to x_2^{i+1} along $\alpha^{-1}\omega_i\alpha$, and concentric to T_1. By repeating this process, we eliminate $\mid \eta_{i+1} \mid$ intersection points. Suppose x_k^j $(j < i)$ is the first point that is not eliminated. Add a tube T starting at x_k^j along $\alpha^\eta \omega_j \alpha^{\eta_j+1} \omega_{j+1} \cdots \omega_i \alpha^{\eta_{i+1}}$ where $\eta = \eta_j + \cdots + \eta_{i+1}$. If $\eta_{i+2} > 0$, we punch holes around the points $x_1^{i+2}, \cdots, x_{\eta_{i+2}}^{i+2}$ to let T go through them to x_1^{i+3}; if $\eta_{i+3} > 0$ we repeat the process until we find a σ with $\eta_\sigma < 0$ and end the tube T at x_1^σ with core

$$\alpha^\eta \omega_j \cdots \omega_{\sigma-1} \alpha^{\eta_\sigma}.$$

Next, we add a new tube T' joining $x_{k-1}{}^{j}$ to $x_2{}^{\sigma}$ and so on. In the process we leave holes around $x_1{}^{i+2}, \cdots, x_{\eta_{i+2}}{}^{i+2}, x_1{}^{i+3}, \cdots$, we now join $x_{\eta_{\sigma-1}}{}^{\sigma-1}$ to $x_{k'}{}^{j,}$ where $x_{k'}{}^{j'}$ is the first point not yet removed. Since $\Sigma\eta_t = 0$ all points are joined by tubes and we obtain an orientable 3-manifold F, obtained from Δ by adding tubes, that is a Seifert manifold for the knot $S^2 \to \partial(K \cup \bigcup h_j^1 \cup \bigcup h_j^2)$. F has free fundamental group in the symbols $b_1, \cdots, b_k$ (one for each added tube) and $\pi_1(\partial(K \cup \bigcup h_j^1) - F)$ is also free in the symbols $\beta_1, \cdots, \beta_r, C_1, \cdots, C_k$ where

$$\nu_0(b_t) = W_t \quad \text{and} \quad \nu_1(b_t) = C_t .$$

Here W_t is a word in $\beta_1, \cdots, \beta_r$ and $C_1, \cdots, C_{t-1}$. This shows that both maps are monomorphisms. Attaching the handles h_j^2 to $\partial(K \cup \bigcup h_j^1) - F$ along the γ_j eliminates the β_j and thus

The knot $S^2 \to \partial(K \cup \bigcup h_j^1 \cup \bigcup h_j^2)$ has a Seifert manifold F such that F and $Y = (S^4 - F)$ have free fundamental group L of the same rank and the maps $\nu_t: L \to L$ are monomorphisms. In that case

$$\tilde{\Pi} = \cdots *_L L *_L L *_L \cdots$$

and by (4) and (5), theorem 2 follows at once.

§4. Further remarks

We can generalize theorems 1 and 2 in two directions:

I. A link is an embedding of m disjoint copies of S^n in S^{n+2}:

$$\ell: mS^n \to S^{n+2}.$$

As usual we consider $\pi = \pi_1(S^{n+2} - \text{Im}(\ell))$ and $H_1(\pi) = Z^m$. If $n \geq 2$, $H_2(\pi) = 0$ and π has weight m. If π has deficiency m then

a) There exists a link $mS^2 \to S^4$ with group π and,

b) $H_q(\pi) = 0$ ($q \geq 2$. (The corresponding statement on the commutator is false.)

II. First of all, observe that if $\Pi = Z_3 \times_\varphi Z$ as described in §1, by the Wang sequence

$$\cdots \to H_q(Z_3) \xrightarrow{\varphi_* - I} H_q(Z_3) \to H_q(\Pi) \to H_{q-1}(Z_3) \to \cdots$$

but $\varphi_* = I$, for $q = 3$, thus

$$H_3(\Pi) = Z_3 \neq 0$$

thus Π cannot have deficiency one by th. 2.

Let $f: S^2 \to S^4$ be a knot and F a minimal (cf. [2; §0]) Seifert manifold. Suppose $\Pi = \pi_1(S^4 - f(S^2))$ and $\tilde{\Pi}$ is finite. With the techniques of [2] one can prove

$$H_3(\Pi) = H_3(\tilde{\Pi})$$

which is obviously false for some groups satisfying (i), (ii) and (iii). If any 2-knot has a minimal Seifert manifold this would show that the class of 2-knot groups is smaller than the class of 3-knot groups. Whether every 2-knot has such minimal Seifert manifolds, I do not know.

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