ON NONSINGULAR BILINEAR MAPS II

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1. Introduction

This paper can be regarded as a continuation of [1] and [2]. Using the Cayley algebra, we construct here new nonsingular bilinear maps. In particular, we are interested in maps with a commutator $xy - yx$, in one of their components. The commutator allows us to have, by restriction of the variables, several other maps. The first such map $R^{16} \times R^{16} \to R^{23}$, was constructed by Lam in [4]. After that, we got $R^{24} \times R^{24} \to R^{39}$ in [1] and $R^{16} \times R^{16+16k} \to R^{23+16k}$ in [2].

Here, we construct $R^{32} \times R^{32} \to R^{56}$, $R^{24} \times R^{32} \to R^{48}$ and $R^{24} \times R^{31} \to R^{47}$ and, from their restrictions, we obtain the nine new maps listed in (4.3). We do not know if the first map, with the commutator condition, can exist in R^{54} or R^{55}. Also, the same question can be considered for the second map in R^{47}.

In section 5 the map $R^{32} \times R^{32} \to R^{54}$, due to Milgram, is constructed using *strong commutators* (see section 2).

Finally, a table of the known nonsingular bilinear maps $R^n \times R^n \to R^t$, for $n \leq 32$, is given.

2. Notation

We will use some of the notation established in [1]. If $x \in K$, where K is the Cayley algebra, define the *trace* and the *norm* of x, respectively, by $t(x) = x + \bar{x}$ and $n(x) = x\bar{x} = \bar{x}x$, where $\bar{x}$ is the conjugate of x.

If $x, y \in K$, we define the *weak commutator* $[x, y]$ as,

$$[x, y] = xy - yx,$$

and the *strong commutator* $\{x, y\}$ as,

$$\{x, y\} = xy - \overline{xy}.$$

We have that,

$$\{x, y\} = 0 \text{ implies } [x, y] = 0.$$

We recall the following fact. Suppose that $[x_1, y_1] = 0$ and that

$$\bar{y}_1 x_2 = x_1 \bar{y}_2 , \tag{2.1}$$

where all the Cayley numbers are different from zero. Then, (2.1) is equivalent to

$$x_2 y_2 n(y_1) = x_1 y_1 n(y_2). \tag{2.2}$$

For a proof see [1; p. 98].

3. The map $R^{32} \times R^{32} \to R^{56}$

In this section we construct a nonsingular bilinear map $R^{32} \times R^{32} \to R^{56}$ with a weak commutator in one of its components.

Let $u = (x_1, x_2, x_3, x_4)$ and $v = (y_1, y_2, y_3, y_4)$, where x_i, y_i for $i = 1, 2, 3, 4$, are Cayley numbers. Set

$$\Psi_1(u, v) = x_1y_1 + x_2y_2 + x_3\bar{y}_4 + \bar{x}_4y_3 \tag{3.1}$$

$$\Psi_2(u, v) = \bar{y}_1x_3 - x_1\bar{y}_3 \tag{3.2}$$

$$\Psi_3(u, v) = \bar{y}_1x_2 - x_1\bar{y}_2 + x_3y_3, \tag{3.3}$$

$$\Psi_4(u, v) = \overline{y_3x_2} + y_2x_3 - x_1y_4 - \overline{x_4y_1} \tag{3.4}$$

$$\Psi_5(u, v) = x_1y_1 - y_1x_1 \tag{3.5}$$

$$\Psi_6(u, v) = \bar{y}_2x_4 - x_2\bar{y}_4 \tag{3.6}$$

$$\Psi_7(u, v) = x_4y_4 - \overline{x_4y_4} \tag{3.7}$$

$$\Psi_8(u, v) = x_2y_2 + x_4y_4 + \overline{x_2y_2} + \overline{x_4y_4} \tag{3.8}$$

$$\Psi_9(u, v) = x_3y_3 + \overline{x_3y_3}. \tag{3.9}$$

Then, define

$$g(u, v) = (\Psi_1(u, v), \cdots, \Psi_9(u, v)).$$

PROPOSITION 3.10. *The bilinear map* $g: R^{32} \times R^{32} \to R^{56}$ *is nonsingular.*

Proof. Assuming that $g(u, v) = 0$ we obtain nine equations $\Psi_k(u, v) = 0$, with $k = 1, \cdots, 9$, that x_i, y_j must satisfy. To prove that g is nonsingular we consider four cases.

First case: if $x_4 \neq 0$ and $y_4 \neq 0$. From (3.7), we have $x_4y_4 = \overline{x_4y_4}$, therefore, $x_4y_4 = y_4x_4$. Then, from (3.8), it follows that

$$2x_4y_4 = -(x_2y_2 + \overline{x_2y_2}) \neq 0. \tag{3.11}$$

Consequently, $x_2 \neq 0$ and $y_2 \neq 0$. From (3.6), using (2.1–2), we get

$$x_4y_4n(y_2) = y_2x_2n(y_4) \tag{3.12}$$

and this implies that $y_2x_2 = \overline{y_2x_2}$, therefore $y_2x_2 = x_2y_2$. Then, from (3.11) we have,

$$x_4y_4 = -x_2y_2$$

and from (3.12), we obtain

$$x_4y_4n(y_2) = x_2y_2n(y_4).$$

Consequently,

$$x_4y_4[n(y_2) + n(y_4)] = 0,$$

and this contradiction settles this case.

Second case: if $x_4 = 0$ and $y_4 = 0$. The system reduces to the following seven equations.

$$x_1y_1 + x_2y_2 = 0 \tag{3.13}$$

(3.14) $$\bar{y}_1x_3 - x_1\bar{y}_3 = 0$$

(3.15) $$\bar{y}_1x_2 - x_1\bar{y}_2 + x_3y_3 = 0$$

(3.16) $$\overline{y_3x_2} + y_2x_3 = 0$$

(3.17) $$x_1y_1 - y_1x_1 = 0$$

(3.18) $$x_2y_2 + \overline{x_2y_2} = 0$$

(3.19) $$x_3y_3 + \overline{x_3y_3} = 0.$$

We need to prove that only the trivial solution is possible. If any x_i or y_i $(i = 1, 2, 3)$ is zero, this is easily set.

Suppose that all the x_i, y_i are different from zero. The components of the nonsingular map given in [1; Th. 3.6] are the lefthand side of (3.13), (3.14), (3.15) and (3.17), plus the expression $\bar{y}_2x_3 - x_2\bar{y}_3$.

Now, it follows from [1; p. 99] that the three equations (3.13–15) plus $[x_1, y_1] = 0$ imply $[x_2, y_2] = 0$. From (3.16) we have $\bar{x}_2\bar{y}_3 = -y_2x_3$ and, using the equivalence (2.1–2), this becomes $x_3y_3n(y_2) = -\overline{x_2y_2}n(y_3)$. Then, with (3.18), we get $x_3y_3n(y_2) = x_2y_2n(y_3)$ and, again using (2.1–2), we obtain $\bar{y}_2x_3 = x_2\bar{y}_3$. Hence, it follows that the system formed by the expressions (3.13–18) is nonsingular. The component (3.9) was not used to settle this case.

Third case: if $x_4 = 0$ and $y_4 \neq 0$. Then, from (3.6), it follows that $x_2 = 0$, and the system of equations is reduced to the following:

$$x_1y_1 + x_3\bar{y}_4 = 0$$

$$\bar{y}_1x_3 - x_1\bar{y}_3 = 0$$

$$x_3y_3 - x_1\bar{y}_2 = 0$$

$$y_2x_3 - x_1y_4 = 0$$

$$x_1y_1 - y_1x_1 = 0$$

$$x_3y_3 + \overline{x_3y_3} = 0.$$

The first five expressions are like the ones of Theorem 4.6 in [1; p. 100]. Consequently, $x_1 = x_3 = 0$. Again, observe that the condition given by the component (3.9) was not used to settle this case.

Fourth case: if $x_4 \neq 0$ and $y_4 = 0$. Then, from (3.6), it follows that $y_2 = 0$, and the system of equations reduces to the following:

(3.20) $$x_1y_1 + \bar{x}_4y_3 = 0$$

(3.21) $$\bar{y}_1x_3 - x_1\bar{y}_3 = 0$$

(3.22) $$\bar{y}_1x_2 + x_3y_3 = 0$$

(3.23) $$y_3x_2 - x_4y_1 = 0$$

(3.24) $$x_1y_1 - y_1x_1 = 0$$

(3.25) $$x_3y_3 + \overline{x_3y_3} = 0.$$

If $x_i = 0$ for some $i = 1, 2, 3$, or if $y_j = 0$ for some $j = 1, 3$, then it follows easily that $v = 0$. Assuming the contrary, from (3.21), (3.24), using the equivalence (2.1–2), we get

(3.26) $$x_1y_1n(y_3) = x_3y_3n(y_1),$$

and substitution of this in (3.20), gives $x_3y_3n(y_1) = -\bar{x}_4y_3n(y_3)$. Now, by cancelling the factor y_3, we obtain that

(3.27) $$x_3n(y_1) = -\bar{x}_4n(y_3).$$

Substitution of this in (3.23), gives $y_3x_2n(y_3) = -\bar{x}_3y_1n(y_1)$, and using $\bar{x}_3y_1 = y_3\bar{x}_1$ that follows from (3.21), we get $y_3x_2n(y_3) = -y_3\bar{x}_1n(y_1)$. Again, by cancelling the factor y_3, we obtain

(3.28) $$x_2n(y_3) = -\bar{x}_1n(y_1).$$

Substitute (3.28) in (3.22) to get $\bar{y}_1\bar{x}_1n(y_1) = x_3y_3n(y_3)$, and then use (3.26) to obtain $\overline{x_1y_1}n(y_1)^2 = x_1y_1n(y_3)^2$. This implies that $n(y_1) = n(y_3)$ and that $x_1y_1 = \overline{x_1y_1}$. On the other hand, from (3.28) and (3.25), it follows that

$$x_1y_1 = x_3y_3 = -\overline{x_3y_3} = -\overline{x_1y_1}.$$

Therefore, $x_1y_1 = 0$. This contradiction ends the proof of (3.10).

4. Maps obtained by restrictions

The map (3.10) gives by suitable restrictions several nonsingular bilinear maps. In particular, we get the following

PROPOSITION 4.1. *There is a nonsingular bilinear map $R^{24} \times R^{32} \to R^{48}$ with a weak commutator in one of its components.*

Proof. Set $x_4 = 0$ in (3.10) and remove the component (3.9). Since (3.9) was not used to establish (3.10) in the cases when $x_4 = 0$, the proposition (4.1) follows.

PROPOSITION 4.2. *There is a nonsingular bilinear map $R^{24} \times R^{31} \to R^{47}$ with a weak commutator in one of its components.*

Proof. Set $x_4 = 0$ and $\bar{y}_2 = -y_2$ in (3.10), and remove the components (3.8) and (3.9). Replace component (3.4) by

$$\Psi_4'(u, v) = x_2\bar{y}_3 + y_2x_3 - x_1y_4.$$

To prove that the map obtained is nonsingular observe that $\Psi_6(u, v) = 0$ implies that either $x_2 = 0$ or $y_4 = 0$. If $x_2 = 0$, the new map agrees with a restriction of (3.10) that does not require (3.9) to establish its nonsingularity.

If $y_4 = 0$, then by replacing $y_2 = -\bar{y}_2$ in $\Psi_4'(u, v)$, we get the $x_2\bar{y}_3 - \bar{y}_2x_3$, and this together with the components (3.1), (3.2), (3.3) and (3.5) constitute the nonsingular map of [1; Th. 3.6]. This ends the proof.

The weak commutator in (3.10), (4.1) and (4.2) allows us to obtain many restrictions as it was shown in [1] and [2]. We only consider those that are not obtainable from [5], [6], and that we regard as new.

PROPOSITION 4.3. *We have the following nonsingular bilinear maps:*

$$R^{21} \times R^{29} \to R^{44} \tag{4.4}$$

$$R^{21} \times R^{28} \to R^{43} \tag{4.5}$$

$$R^{19} \times R^{27} \to R^{42} \tag{4.6}$$

$$R^{19} \times R^{26} \to R^{41} \tag{4.7}$$

$$R^{19} \times R^{31} \to R^{46} \tag{4.8}$$

$$R^{19} \times R^{30} \to R^{45} \tag{4.9}$$

$$R^{23} \times R^{27} \to R^{46} \tag{4.10}$$

$$R^{23} \times R^{26} \to R^{45} \tag{4.11}$$

$$R^{29} \times R^{29} \to R^{52}. \tag{4.12}$$

Proof. As in [1; p. 99], let r_i, z_i and q_i denote, respectively, any real, any complex and any quaternion number. The first eight maps are obtained by restricting x_1, y_1, respectively in (4.1) and (4.2), as follows. For (4.4–5), $x_1 = (r_1, q_1)$, $y_1 = (r_2, q_2)$; for (4.6–7), $x_1 = (r_1, z_1), y_1 = (r_2, z_2)$; for (4.8–9), $x_1 = (r_1, z_1)$, $y_1 = ((r_2, z_2), q_2)$; for (4.10–11), $x_1 = ((r_1, z_1), q_1)$, $y_1 = (r_2, q_2)$. The map (4.12) is obtained from (3.10), with $x_1 = (r_1, q_1)$, $y_1 = (r_2, q_2)$.

5. The map $R^{32} \times R^{32} \to R^{54}$

Milgram constructs in [6] a map $R^{32} \times R^{32} \to R^{54}$. With some modifications this map can be presented as follows. Let u, v be as in section 3. Using the notation of section 2, set

$$\begin{aligned}
\Phi_1(u, v) &= \textstyle\sum_{i=1}^{4} t(x_i y_i)\\
\Phi_2(u, v) &= \{x_1, y_1\}\\
\Phi_3(u, v) &= x_2 y_1 - \overline{x_1 y_2}\\
\Phi_4(u, v) &= t(x_3 y_1 - x_1 y_3 + x_4 y_2 - x_2 y_4)\\
\Phi_5(u, v) &= \{x_3, y_1\} + \{x_1, y_3\} + \{x_2, y_2\}\\
\Phi_6(u, v) &= \{x_4, y_2\} + \{x_2, y_4\} + \{x_3, y_3\}\\
\Phi_7(u, v) &= x_4 y_1 - \overline{x_1 y_4} + x_2 y_3 - \overline{x_3 x_2}\\
\Phi_8(u, v) &= x_4 y_3 - \overline{x_3 y_4}\\
\Phi_9(u, v) &= \{x_4, y_4\}.
\end{aligned}$$

Then, define

$$f(u, v) = (\Phi_1(u, v), \cdots, \Phi_9(u, v)).$$

PROPOSITION 5.1. *The bilinear map* $f: R^{32} \times R^{32} \to R^{54}$ *is nonsingular.*

We leave the proof as an exercise for the interested reader.

PROPOSITION 5.2. *The map f induces the following nonsingular bilinear maps:*

$$R^{28} \times R^{28} \to R^{50} \tag{5.3}$$

$$R^{26} \times R^{26} \to R^{48} \tag{5.4}$$

$$R^{25} \times R^{25} \to R^{47}. \tag{5.5}$$

Proof. To obtain these maps restrict in the strong commutator

$$\Phi_9(u, v) = x_4 y_4 - \overline{x_4 y_4},$$

x_4 and y_4 to be, respectively, quaternion, complex and real numbers.

The maps (5.3), (5.4) can also be obtained from [6; Cor. 4][1]. However, the map (5.5) has not been made explicit there.

6. A table

The real projective space P^{n-1} has an immersion in R^{t-1} if and only if there exists a nonsingular skew-linear map $R^n \times R^n \to R^t$, with $t > n$. The existence of a bilinear map seems to be a stronger condition than just the existence of an immersion. In [3; p. 91] a table of immersions of P^n for $n < 40$ was given. The following is a table of the first few nonsingular bilinear maps, known to exist.

Nonsingular bilinear maps $R^n \times R^n \to R^t$

n	t	n	t	n	t
1	1	12	17	23	39
2	2	13	19	24	39
3	4	14	23	*25	47
4	4	15	23	*26	48
5	8	16	23	*27	50
6	8	17	32	*28	50
7	8	18	32	*29	52
8	8	19	33	*30	54
9	16	*20	35	*31	54
10	16	21	35	32	54
11	17	22	39		

The cases without * are the best possible, in the sense that for the given n the $t = \varphi(n)$ is the minimal. All the information necessary to construct the maps of this table is contained in [1] and [4], together with the results of this paper.

[1] It seems to be a misprint in [6; Th. 3] so that the exponent $n + m + 1 - [\alpha(n) + \alpha(n - m) + t]$ should be $n + m + 1 - [\alpha(n) - \alpha(n - m) + t]$.

Finally, since we have the immersions $P^{19} \subseteq R^{32}$ and $P^{24} \subseteq R^{39}$, for the first two unsettled cases we ask: Can we have $\varphi(20) = 33$ and $\varphi(25) = 40$?

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