INERTIA GROUPS OF HOMOTOPY PROJECTIVE SPACES

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Let M^m be a closed smooth oriented manifold, and let θ^m be the group of h-corbordism classes of homotopy m-spheres of [Kervaire-Milnor]. Let $I(M^m) \subset \theta^m$ be the set of $\Sigma^m \in \theta^m$ such that the connected sum $M \# \Sigma$ is diffeomorphic to M by an orientation preserving diffeomorphism. Then $I(M)$ is a subgroup of θ^m and if $bP_{m+1} \subset \theta^m$ is the subgroup of Σ which bound parallelizable manifolds, let $I_0(M) = I(M) \cap bP_{m+1}$.

In [Browder, A], we studied $I_0(M^m)$ for $m = 4k - 1$, in particular when M^m is a π-manifold. In this note we use these methods to study the case where $m = 4k - 1$ and M is a homotopy real projective space, proving the result quoted in [Lopez de Medrano, V7].

THEOREM *Let M^m be a homotopy real projective space of dimension $m = 4k - 1$, $k > 1$. Then $I_0(M) = 0$.*

We begin with a result of [Olum]:

LEMMA 1 *For m odd, any orientation preserving homotopy equivalence $h: P^m \to P^m$ (real projective space) is homotopic to the identity.*

Following [Browder, A.], define $Q^{4k} = (M \times [0, 1] \natural W)/g$ where $\partial W = \Sigma \in I_0(M)$, W parallelizable, $\natural$ denotes connected sum along the boundary $M \times 1$, $g: M \times 0 \to (M \times 1) \# \partial W$ an orientation preserving diffeomorphism, so that Q is the identification space of $M \times [0, 1] \natural W$ by the diffeomorphism g on the two boundary components.

LEMMA 2 *Q/W is a fibre bundle over S^1 with fibre M.*

This is [Browder, A (2.1)].

From now on let M be homotopy equivalent to P^m, m odd. Combining lemmas 1 and 2 we get:

LEMMA 3 *Q/W is homotopy equivalent to $M \times S^1$.*

For a bundle over S^1 with fibre M is determined by an automorphism of M and by Lemma 1 this is homotopic to the identity, so the lemma follows.

LEMMA 4 *There is a linear bundle ξ^k over $M \times S^1$ such that $c^*(\xi) = \nu_Q$, the stable normal bundle of $Q^{m+1} \subset S^{m+k+1}$, where $c: Q \to Q/W \cong M \times S^1$ is the collapsing map.*

This follows easily since W is parallelizable.

LEMMA 5 *ξ is a fibre homotopy equivalent to $\nu_{M \times S^1}$.*

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Proof: Let $\hat{c}: T(\nu_Q) \to T(\xi)$ be the bundle map covering c. Then the composite $S^{m+1+k} \xrightarrow{\alpha} T(\nu_Q) \xrightarrow{c} T(\xi)$ is of degree 1 on the top homology, where α is the natural collapsing map. It then follows from [Atiyah] that ξ is the fibre homotopy equivalent to $\nu_{M \times S^1}$. (See also [Spivak] or [Browder, S I§4]).

LEMMA 6 *Let* $d: M^m \times S^1 \to S^{4k}$ $(m = 4k - 1)$ *be a map of degree* 1. *There is a linear bundle* η *over* S^{4k} *such that* $[\xi] = [\nu_{M \times S^1}] + d^*[\eta]$ *in* $KO(M \times S^1)$.

Proof: Since $\xi \mid M \times \frac{1}{2} = \nu_M + \epsilon^1 = \nu_Q \mid M \times \frac{1}{2}$, it follows that if $\gamma = (\xi - \nu_{M \times S^1})$, $\gamma \mid M \times \frac{1}{2}$ and $\gamma \mid (pt) \times S^1$ are trivial so that $\gamma = r^*(\gamma')$, where γ' is a linear bundle over $\Sigma M = M \times S^1/(M \times \frac{1}{2}) \cup (\text{pt.}) \times S^1)$ and γ is fibre homotopy trivial. By [Fujii], $KO(\Sigma P^{m-1}) = Z_2$ and the generator has non-zero Stiefel-Whitney classes. Since γ' is fibre homotopy trivial it follows that $[\gamma' \mid \Sigma P^{m-1}] = 0$ so that $[\gamma'] = d'^*[\eta]$ for $[\eta] \in KO(S^{4k})$, where $d': \Sigma M \cong \Sigma P^m \to \Sigma P^m / P^{m-1} \cong S^{4k}$. Hence $[\xi] = [\nu_{M \times S^1}] + r^* d'^*[\eta] = [\nu_{M \times S^1}] + d^*[\eta]$, since $d'r = d$.

Recall from [Milnor-Kervaire] the following:

LEMMA 7. *If* η *is a linear bundle over* S^{4k} *then* $p_k(\eta)$ *is divisible by* $(2k - 1)!\, a_k$, *where* $a_k = 2$ *for* k *odd*, $a_k = 1$ *for* k *even*.

Now we have, since $P(\nu_{M \times S^1}) = 1$

$$P(Q)^{-1} = P(\nu_Q) = P(c^*\xi) = c^*P(\xi) = c^*P(\nu_{M \times S^1} + d^*\eta) =$$
$$c^*(P(\nu_{M \times S^1})P(d^*\eta)) = c^*(P(d^*\eta)) = c^*d^*P(\eta),$$

so that $p_i(Q) = 0$ for $i < k$ and

$$p_k[Q] = (c^*d^*p_k(\eta)^{-1})[Q] = p_k(-\eta)(d_*c_*[Q] = p_k(-\eta)[S^{4k}].$$

By Lemma 7, $(2k - 1)!a_k$ divides $p_k(-\eta)$, so that $(2k - 1)!\,a_k$ divides the Pontrjagin number $p_k[Q]$, while all other Pontrjagin numbers of Q are zero.

Now P^{4k-1} is a Spin-manifold since $w_1(P^{4k-1}) = w_2(P^{4k-1}) = 0$. Since w_1, w_2 are homotopy invariants it follows that M^m and $M^m \times S^1$ are Spin-manifolds so that $w_1(\xi) = w_2(\xi) = 0$ by Lemma 5 so that $w_1(\nu_Q) = w_2(\nu_Q) = 0$ by Lemma 4 and Q is also a Spin-manifold.

Hence, from [Atiyah-Hirzebruch]:

$$\hat{A}[Q] = \frac{B_k}{2(2k)!} p_k[Q] = \frac{B_k}{2(2k)!} (2k - 1)!\, a_k y$$

is an integer, even for k odd (when $a_k = 2$) so that y is a multiple of denominator $(B_k/4k)$, where $p_k[Q] = (2k - 1)!a_k y$. From the verification of the Adams conjecture [Quillen] [Sullivan] [Mahowald] and [Friedlander], denominator $\left(\frac{B_k}{4k}\right) = d_k$ is the order of im $J \subset \pi_{4k}{}^S$, and $y = z\, d_k$.

From the Index Theorem of [Hirzebruch] it follows that

$$\begin{aligned}\sigma(Q) &= 2^{2k}(2^{2k-1} - 1)B_k p_k[Q]/(2k)! \\ &= 2^{2k}(2^{2k-1})B_k a_k (2k-1)!\, d_k z/(2k)! \\ &= 2^{2k-1}(2^{2k-1} - 1)B_k d_k a_k z/k = z\sigma_k\end{aligned}$$

where $\sigma_k = 8(\text{order } bP_{4k})$ from [Kervaire-Milnor, page 531].

From [Browder, A (2.6)] we get that $\sigma(W) = \sigma(Q) = z\sigma_k$, so that ∂W is h-corbordant to S^{4k-1} by [Kervaire-Milnor, (7.5)] and we conclude that $I_0(M) = 0$.

Remarks

1. M^m, the 2-fold covering of M is a homotopy sphere, and if m is odd, the 2-fold covering of $M \# \Sigma$ is $\tilde{M} \# \Sigma \# \Sigma$. Therefore if $\Sigma \in I(M)$, then $2\Sigma \in I(\tilde{M}) = 0$, since the homotopy spheres form a group under connected sum. Hence all our efforts were really necessary only to check the element of order 2 in bP_{4k}.

2. Using the verification of the Adams conjecture alluded to above and utilizing the techniques of [Brumfiel], the main result of [Browder, A] may be made more precise: *If M^{4k-1} is a π-manifold, then $I_0(M) = 0$.*

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