A CONNECTED COUNTABLE HAUSDORFF SPACE, S_α FOR EVERY COUNTABLE ORDINAL α

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We say that a topological space X is S_α for some ordinal α if, given distinct points $x, y \in X$, we can find open neighborhoods U_β $(\beta < \alpha)$ of x and V_β $(\beta < \alpha)$ of y such that (1) $U_0 \cap V_0 = \emptyset$; and (2) for any $\delta < \gamma < \alpha$, $\bar{U}_\gamma \subset U_\delta$ and $\bar{V}_\gamma \subset V_\delta$. The definition of S_α was originally given by Porter and Votaw [2].

In the present paper we constuct a countable connected locally connected topological space X which is S_α for every countable ordinal α. In [1], Jones and Stone pose the question of existence of a countable connected space which is P_α for every countable ordinal α. Since S_α implies $P_{\alpha+1}$, X answers that question affirmatively.

The model spaces $\mathbf{M}_\alpha$ and $\mathbf{N}_\alpha$. For any ordinal $\alpha = \lambda + n$, where λ is 0 or a limit ordinal and n is an integer, let $e(\alpha) = \lambda + 2n$, and let M_α be the set of all ordinals $\beta \leqslant e(\alpha)$, with the topology generated by the sets $A_\gamma = \{0 \leqslant \beta < e(\gamma)\}$ and $B_\gamma = \{e(\gamma) < \beta \leqslant e(\alpha)\}$ for all $\gamma \leqslant \alpha$. Note that for any $0 \leqslant \delta < \gamma \leqslant \alpha$, $\bar{B}\gamma = \mathcal{C}A_\gamma \subset B_\delta$ (where $\mathcal{C}$ denotes complementation). Let N_α be the quotient space obtained from $M_\alpha \times \{0, 1\}$ by identifying $(0, 0)$ with $(0, 1)$; this identified point we call "0." Let N_α be partially ordered as follows:

(i) If $0 < \delta < \gamma \leqslant \alpha$, $(\delta, 0) < (\gamma, 0)$ and $(\delta, 1) < (\gamma, 1)$.
(ii) For any $0 < \gamma \leqslant \alpha$, $0 < (\gamma, 0)$ and $0 < (\gamma, 1)$.

Note that M_α and N_α are both connected and locally connected, and that any connected subset of N_α has a least element.

In a certain sense, N_α, though not even Hausdorff, is a model for the property S_α; *i.e.*, the two points $(\alpha, 0)$ and $(\alpha, 1)$ can be separated by appropriately nested sets. For each $0 \leqslant \beta < \alpha$, let $C_\beta = B_\beta \times \{0\}$, and let $D_\beta = B_\beta \times \{1\}$, neighborhoods of $(\alpha, 0)$ and $(\alpha, 1)$, respectively; then (1) $C_0 \cap D_0 = \emptyset$; and (2) for any $\delta < \gamma \leqslant \alpha$, $\bar{C}_\gamma \subset C_\delta$ and $\bar{D}_\gamma \subset D_\delta$.

Lemma 1: *For any ordinal $\alpha < \Omega$, let X_α be any countable (finite or infinite) set, and let a_α, $b_\alpha \in X_\alpha$ be specified. Then there exists a countable set $S \subset \prod_{\alpha<\Omega} X_\alpha$ such that*

(i) *For each finite set F of countable ordinals, the projection $P_F: S \to \prod_{\alpha\in F} X_\alpha$ is onto.*

(ii) *For any $\alpha < \Omega$ and for any $x, y \in S$, there exists a countable ordinal $\beta \geqslant \alpha$ such that $\{p_\beta x, p_\beta y\} = \{a_\beta, b_\beta\}$, where $p_\beta: S \to X_\beta$ is the projection.*

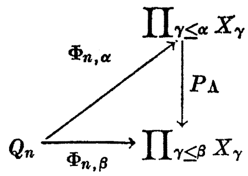
Proof: It suffices to consider only the case where each X_α is infinite; we shall make that assumption hereafter. Henceforth, for any $\beta \in G \subset H$, G and H sets of ordinals less than Ω, let $P_G: \prod_{\alpha\in H} X_\alpha \to \prod_{\alpha\in G} X_\alpha$ and $p_\beta: \prod_{\alpha\in G} \to \mathrm{X}_\beta$ denote the obvious projections.

For each positive integer n, let Q_n be an infinite set of positive integers; let the Q_n be chosen so that Q_i and Q_j are disjoint if $i \neq j$, and such that every positive integer lies in one of the Q_n. Also, for each ordered pair of positive integers (i, j), where $i < j$, let $\Omega_{i,j}$ be any *uncountable* set of countably infinite ordinals; we insist that $\Omega_{i,j}$ and $\Omega_{p,q}$ are disjoint if $(i, j) \neq (p, q)$. For each integer $n \geqslant 1$, we shall inductively, on α, define a function:

$$\Phi_{n,\alpha}: Q_n \to \prod_{\beta \leq \alpha} X_\beta$$

such that:

(1) For each $\beta < \alpha$, the following diagram is commutative:



where $\Lambda = \{\gamma \leqslant \beta\}$.

(2) For each set F of exactly n distinct ordinals less than or equal to α, the composition $P_F \circ \Phi_{n,\alpha}: Q_n \to \prod_{J \in F} X_J$ is onto.

(3) If $\alpha \in \Omega_{i,j}$ for some (i, j), then $i \in Q_n \Rightarrow p_\alpha \Phi_{n,\alpha}(i) = a_\alpha$ and $j \in Q_n \Rightarrow p_\alpha \Phi_{n,\alpha}(j) = b_\beta$.

We let $\Phi_{n,n-1}: Q_n \to \prod_{\alpha \leq n-1} X_\alpha$ be any onto function. For any $0 \leqslant k < n - 1$, let $\Phi_{n,k} = P_{k+1} \circ \Phi_{n,n-1}$. Properties (1) and (2) are obviously satisfied for all $\alpha \leqslant n - 1$, while (3) is fulfilled vacuously since $\Omega_{i,j}$ contains no finite ordinals.

Suppose now that $n \leqslant \alpha < \Omega$, and $\Phi_{n,\beta}$ has been defined for all $\beta < \alpha$, satisfying (1), (2), and (3) above. To define $\Phi_{n,\alpha}$, it is necessary and sufficient to specify $p_\alpha \Phi_{n,\alpha}(k)$ for all $k \in Q_n$; property (1) will then be satisfied automatically.

If $\alpha \in \Omega_{i,j}$ and $i \in Q_n$, let $p_\alpha \Phi_{n,\alpha}(i) = a_\alpha$. If $\alpha \in \Omega_{i,j}$ and $j \in Q_n$, let $p_\alpha \Phi_{n,\alpha}(j) = b_\alpha$. Thus (3) is assured.

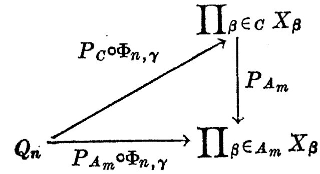
Henceforth, let $Q_n' = Q_n - \{i, j\}$ if $\alpha \in \Omega_{i,j}$ for some (i, j); otherwise let $Q_n' = Q_n$. Let $\mathfrak{A}$ be the set of all ordered pairs of the form (A, x) where A is a set of ordinals less than α with exactly $n - 1$ elements, and where $x \in \prod_{\beta \in A'} X_{\beta'}$, where $A' = A \cap \{\alpha\}$. Let $(A_1, x_1), (A_2, x_2), \cdots$ be a specific denumeration of $\mathfrak{A}$. We shall inductively, on m, define $z_m \in Q_n'$ for each integer $m \geqslant 1$, such that:

(4) $P_{A_m} \Phi_{n,\beta} z_m = P_{A_m} x_m$ for all $\max A_m \leqslant \beta < \alpha$

(5) $z_m \neq z_k$ for all $k < m$.

For any $m \geqslant 1$, suppose that $z_k \in Q_n'$ has been chosen for all $1 \leqslant k < m$, satisfying (4) and (5). It is of course only necessary that (4) be satisfied for $\beta = \beta_m = \max A_m$. Pick an ordinal $\delta < \alpha$ such that $\delta \notin A_m$. This is always possible since α

has at least n predecessors. Let $C = A_m \cup \{\delta\}$. Let $\gamma = \max C$. We have a commutative diagram:



where all functions are onto. Note that ${P_{A_m}}^{-1} y$ is in one-to-one correspondence with X_γ, hence is infinite, for any $y \in \prod_{\beta \in A_m} X_\beta$. We now choose z_m to be any element of $Q_n' \cap (P_{A_m} \circ \Phi_{n,\gamma})^{-1} P_{A_m} x_m$ such that $z_m \neq z_k$ for any $1 \leqslant k < m$. This is always possible since only finitely many possibilities have been eliminated. Once all z_m are given, we define $p_\alpha \Phi_{n,\alpha}(z_m) = p_\alpha x_m$ for all m. If $i \in Q_n'$, and if $i \neq z_m$ for any m, let $p_\alpha \Phi_{n,\alpha}(i)$ be any element of X_α whatsoever.

We need only check that (2) is satisfied. Suppose that F is a set of ordinals less than or equal to α, and that F has exactly n elements. Let $\beta = \max F$. If $\beta < \alpha$, we are done, by the inductive hypothesis. If $\beta = \alpha$, let $x \in \prod_{\gamma \in F} X_\gamma$. Then $(F - \{\alpha\}, x) = (A_m, x_m)$ for some integer $m \geqslant 1$; $(P_F \circ \Phi_{n,\alpha})(z_m) = x$, and we are done.

For each n, let $\Phi_n : Q_n \to \prod_{\beta < \Omega} X_\beta$ be the unique function such that $P_{\alpha+1} \circ \Phi_n = \Phi_{n,\alpha}$ for all $\alpha < \Omega$, and let $S_n = \Phi_n(Q_n) \subset \prod_{\beta<\Omega} X_\beta$. Let $S = \bigcup_{n=1}^{\infty} S_n$. If F is any finite set of ordinals less than Ω, $P_F(S) \supset P_F(S_n) = \prod_{\beta \in F} X_\beta$, where n is the cardinality of F. Hence (i) is fulfilled. And if $\beta \in \Omega_{i,j}$; $p_\beta(\Phi(i)) = a_\beta$ and $p_\beta(\Phi(j)) = b_\beta$, where $\Phi = \bigcup_{n=1}^{\infty} \Phi_n$, hence (ii) is fulfilled. This concludes the proof of Lemma 1.

The Space X. Let $X \subset \prod_{\alpha<\Omega} N_\alpha$ be any countable subset such that (1) for F any finite set of countable ordinals, the projection $P_F : X \to \prod_{\alpha \in F} N_\alpha$ is onto; and (2) for any $x, y \in X$ and any $\beta < \Omega$, there exists countable $\alpha \geqslant \beta$ such that $p_\alpha(x) = (\alpha, 0)$ and $p_\alpha(y) = (\alpha, 1)$. Existence of such a set is guaranteed by Lemma 1.

We show that X is S_α for every countable α. Let $x, y \in X$. Without loss of generality, $p_\alpha(x) = (\alpha, 0)$ and $p_\alpha(y) = (\alpha, 1)$. For every $\beta < \alpha$, let $U_\beta = {p_\alpha}^{-1} C_\beta$ and $V_\beta = {p_\alpha}^{-1} D_\beta$. Conditions (1) and (2) in the first paragraph of this paper are then satisfied.

Finally, from Lemma 2, below, it follows that X is connected and locally connected; the latter since each N_α is locally connected.

LEMMA 2: *For each $\alpha < \Omega$, let Y_α be any connected subset of N_α, such that $Y_\alpha = N_\alpha$ for all but finitely many choices of α. Let $Y = X \cap \prod_{\alpha<\Omega} Y_\alpha$. Then Y is connected.*

Proof: For each α, let $m_\alpha \in Y_\alpha$ be the least element. We shall prove a lemma:

LEMMA 3: *Let F be any finite set of countable ordinals, and let $\alpha \in F$. For each $\beta \in F$, let $x_\beta \in N_\beta$ be given, such that $x_\alpha = m_\alpha$. Suppose that $U \subset Y$ is open and* *closed (in Y), and suppose that for all $y \in Y$, $y \in U$ provided $p_\beta y = x_\beta$ for all $\beta \in F$. Then, for all $y \in Y$, $y \in U$ provided $p_\beta y = x_\beta$ for all $\beta \in F - \{\alpha\}$.*

Proof: We prove this lemma by induction on $p_\alpha y$. Clearly, if $p_\alpha y$ is required to be m_α, there is no problem. Suppose then, for some $a \in N_\alpha$, the statement of the lemma holds for all y such that $m_\alpha \leqslant p_\alpha y < a$. We then show that it holds if $p_\alpha y = a$.

Case I: a odd. That is, $\{a\}$ is open. Let $b \in Y_\alpha$ be the immediate predecessor of a. Every neighborhood of b must then contain a. Let W be an arbitrary neighborhood of y. Pick open sets $\{V_\beta \subset Y_\beta\}_{\beta<\Omega}$ such that, for some finite G, $V_\beta = N_\beta$ for all $\beta \notin (F \cup G)$, and such that $y \in \prod_{\beta<\Omega} V_\beta \subset W$. Pick $y' \in Y$ such that $p_\alpha y' = b$ and $p_\beta y' = p_\beta y$ for all $\beta \in (F \cup G) - \{\alpha\}$. By hypothesis, $y' \in U$. Pick open sets $\{U_\beta \subset Y_\beta\}_{\beta<\Omega}$ such that, for some finite H, $U_\beta = N_\beta$ for all $\beta \notin (F \cup G \cup H)$, and such that $y' \in \prod_{\beta<\Omega} U_\beta \subset U$. Now $a \in U_\alpha$. Pick $y'' \in Y$ such that $p_\alpha y'' = a$ and $p_\beta y'' = p_\beta y'$ for all $\beta \in (F \cup G \cup H) - \{\alpha\}$. Then $y'' \in U \cap W$. It follows that $y \in \bar{U} = U$.

Case II: a even. That is, $\{a\}$ is closed. Then every neighborhood of a contains a predecessor of a. Let W be an arbitrary neighborhood of y. Pick a finite set G and open sets $\{V_\beta \subset Y_\beta\}_{\beta<\Omega}$ such that $V_\beta = N_\beta$ for all $\beta \notin G$, and $y \in \prod_{\beta<\Omega} V_\beta \subset W$. Pick $b \in N_\alpha$ such that $b < a$ and $b \in V_\alpha$, and pick $y' \in Y$ such that $p_\alpha y' = b$ and $p_\beta y' = p_\beta y$ for all $\beta \in (F \cup G) - \{\alpha\}$. By hypothesis, $y' \in U$. On the other hand, $y' \in W$. Thus $y \in \bar{U} = U$. This completes the proof of Lemma 3.

Returning to the proof of Lemma 2, we suppose that Y is disconnected. Let U and V be non-empty disjoint open sets such that $Y = U \cup V$, Pick $u \in U$. Pick open sets $\{U_\beta \subset Y_\beta\}_{\beta<\Omega}$ and $\{V_\beta \subset Y_\beta\}_{\beta<\Omega}$ such that, for some finite set F, $U_\beta = V_\beta = N_\beta$ for all $\beta \notin F$, and such that $u \in \prod_{\beta<\Omega} U_\beta \subset U$ and $v \in \prod_{\beta<\Omega} V_\beta \subset V$. Pick $w \in Y$ such that $p_\beta w = m_\beta$ for all $\beta \in F$. Without loss of generality, $w \in U$. Pick open sets $\{W_\beta \subset Y_\beta\}_{\beta<\Omega}$ such that $w \in \prod_{\beta<\Omega} W_\beta \subset U$, and such that, for some finite set G, $W_\beta = N_\beta$ for all $\beta \notin G$. Let $F = (\alpha_1, \alpha_2, \cdots \alpha_n)$. For each $0 \leqslant i \leqslant n$, we inductively prove the statement:

(R_i) For any $y \in Y$, $y \in U$ provided (1) $p_\beta y \in W_\beta$ for all $\beta \in G - F$, and (2) $p_{\alpha_j} y = m_{\alpha_j}$ for all $i < j \leqslant n$.

R_0 is implied by the statement $\prod_{\beta<\Omega} W_\beta \subset U$, while $R_i \Rightarrow R_{i+1}$ by Lemma 2, for all $0 \leqslant i < n$. Let $y \in Y$ be any element where $p_\beta y = p_\beta v$ for all $\beta \in F$, and $p_\beta y = p_\beta w$ for all $\beta \in G - F$. *A priori*, $y \in \prod_{\beta<\Omega} V_\beta \subset V$, while $R_n \Rightarrow y \in U$: contradiction.

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References

[1] F. B. Jones and Stone, A. H. *Countable locally connected Urysohn spaces*, Coll. Math. **22** (1971), 239–44.

[2] J. R. Porter and Votaw, Ch., $S(\alpha)$ *spaces and regular Hausdorff extensions* (*to appear*).