ON FAMILIES OF HYPERSURFACES WITH WEAKLY POSITIVE NORMAL BUNDLE

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0. Introduction

In this paper we extend a result of H. Rossi on the projective embedding of a neighborhood of a hypersurface with weakly positive normal bundle. We will prove the theorem with parameters.

Definitions. Our analytic spaces will be ringed spaces as in [5]. Let Y be a compact analytic space, with $L \to Y$ a line bundle; L is weakly positive if there is a relatively compact 1-pseudoconcave neighborhood of the zero-section of L. For X an analytic space, $Y \subset X$ a compact subspace, let $\mathcal{I}$ be the ideal sheaf of Y in X, and let E be the associated line bundle (where if $\mathcal{E}$ is the sheaf of germs of sections of E, then $\mathcal{E} = \mathcal{I}^{-1}$). Let $N = E \mid Y$ be the normal bundle of Y in X; then Y is of algebraic codimension 1 in X if $\mathfrak{N} = (\mathcal{I}/\mathcal{I}^2)^{-1}$.

For X, D analytic spaces, we say $f: X \to D$ is an analytic family if for each point $x \in X$ there is a neighborhood U of x and charts $g: U \to C^m \times C^n$ and $g': f(U) \to C^n$ such that

$$\begin{array}{ccc} U & \xrightarrow{g} & C^m \times C^n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \text{proj}} \\ f(U) & \xrightarrow{g'} & C^n \end{array}$$

is commutative. Let $\text{dih}_x(X)$ be the homological dimension of X at x, as defined in [1], p. 197.

MAIN THEOREM. *Let $f: X \to D$ be an analytic family, with $Y \subset X$ a subfamily. Assume that X is irreducible and paracompact, that D is a manifold, and that, for each $d \in D$, the following hold:*

(1) *X_d is irreducible of pure dimension $k > 2$;*

(2) *Y_d is compact and of algebraic codimension 1 in X_d;*

(3) *the normal bundle N_d of Y_d in X_d is weakly positive;*

(4) *$dih_Y(X) - 2\dim(D) > 2$, where $dih_Y(X) = \min\{dih_x(X): x \in Y\}$.*

Then there exists a neighborhood W of Y in X, an analytic family $f_v: V \to D$ with each fiber a projective variety of pure dimension k, and a fiber-preserving injection $g: W \to V$ such that $g(Y)$ is the family of hyperplane sections of V. (In other words, $V - g(Y)$ is a family of affine varieties).

The theorem for D a point is Theorem 3, p. 250 of [6]; we follow the idea of

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the proof given there. The two key theorems needed for the generalization are the following, the first proved by Siu in [8], the second proved by Rossi in [7].

THEOREM 1. *Let $f: X \to D$ be a 1-pseudoconcave family with exhaustion function φ and concavity bounds r_*, $r_\#$. Suppose D is a complex manifold, and let $\mathcal{S}$ be a π-flat coherent analytic sheaf on X; then for $r \in (r_*, r_\#)$, the direct image sheaves $R^i f_*(\mathcal{S} \mid X^r)$ are coherent at $d \in D$ for $i < \min\{\operatorname{dih}_x(\mathcal{S}) : x \in X_d \cap \varphi^{-1}(r)\} - 2 \dim(D) - 1$.*

See [8] for definitions and notations. This theorem extends finiteness of cohomology results found in [1].

THEOREM 2. *Let X be an analytic space such that $\mathcal{O}(X)$ separates points of X. Let $\varphi: X \to M$ be a holomorphic map of X into a complex manifold M. Let U be an open set in M such that all holomorphic functions on U extend to M. Suppose V is a closed sub-variety of $\varphi^{-1}(U)$ such that $\varphi: V \to U$ is light, proper, and surjective. Then there is a closed subvariety $\tilde{V}$ of X such that $\tilde{V} \cap \varphi^{-1}(U) = V$.* (Theorem 2.7, p. 568 of [7].)

Note: φ is light if $\varphi^{-1}(m)$ is a finite set of points for each $m \in M$; φ is proper if $\varphi^{-1}(K)$ is compact whenever $K \subset M$ is compact.

Our assumption (4) is the only assumption which must be added to extend the result on one fiber; it is necessary for the application of theorem 1, as is the hypothesis that D is a manifold. A weakening of these hypotheses for theorem 1 would produce the corresponding extension of the main theorem of this paper.

In section 1 we will construct an injection into a family of projective spaces over small open sets in D, using Theorem 1; in section 2 we will show the image can be extended to a family of projective varieties, using theorem 2. Finally, we show in section 3 that all the local families fit together to form a family of varieties over all of D.

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1. Construction of an embedding

PROPOSITION 1. *Let $f: X \to D$ be an analytic family, with subfamily $f: Y \to D$ satisfying the assumptions of the main theorem. Assume further that*

(5) *$f: X \to D$ is a 1-pseudoconcave map, with exhaustion function φ and concavity bound $(r_*, r_\#)$;*

(6) *there is an $r_0 > r_*$ such that $Y \subset \varphi^{-1}(r_0, \infty]$.*

Then for each $d \in D$ there is a neighborhood B of d, a neighborhood W of $Y \mid B$ in $X \mid B$, and a fiber-preserving injection $g: W \to \boldsymbol{P}^m \times B$ such that $g(Y \mid B)$ is contained in $H \times B$, where H is the hyperplane at ∞ of $\boldsymbol{P}^m$.

Remark. After proving the proposition we will show that hypotheses (5) and (6) follow from the others.

Proof. Restrict X to some smaller neighborhood of Y such that $\operatorname{dih}_{X_d}(X) = \operatorname{dih}_{Y_d}(X)$ for all $d \in D$, where $\operatorname{dih}_{X_d}(X) = \min\{\operatorname{dih}_x(X) : x \in X_d\}$.

Let $\mathcal{I}$ be the ideal sheaf of Y in X, let E be the associated line bundle, and let $\mathcal{E}$ be the sheaf of germs of sections of E; then $\mathcal{E} = \mathcal{I}^{-1}$. As $N = E \mid Y$, we have $\mathfrak{N} = \mathcal{E}/\mathcal{I}\cdot\mathcal{E}$. Tensor the exact sequence

$$0 \to \mathcal{I}^2 \to \mathcal{O} \to \mathcal{O}/\mathcal{I}^2 \to 0$$

by the locally free sheaf $\mathcal{E}^s$; let $Q^s = \mathcal{E}^s \otimes \mathcal{O}/\mathcal{I}^2 \cong \mathfrak{N}^s \oplus \mathfrak{N}^{s-1}$. Then we can write the resulting sequence as

$$0 \to \mathcal{E}^{s-2} \xrightarrow{h} \mathcal{E}^s \to Q^s \to 0,$$

from which we get the long exact sequence of direct image sheaves

$$0 \to R^0 f_*(\mathcal{E}^{s-2}) \xrightarrow{h_*} R^0 f_*(\mathcal{E}^s) \to R^0 f_*(Q^s) \to$$

$$(*) \qquad R^1 f_*(\mathcal{E}^{s-2}) \xrightarrow{h_*} R^1 f_*(\mathcal{E}^s) \to R^1 f_*(Q^s) \to \cdots .$$

The sheaf $\mathfrak{N}_d^{\,s}$ of germs of sections of $N_d^{\,s}$ is supported on the compact set Y_d; thus, by Satz 2, p. 343 of [3], there exists an integer s_1 such that for all $s \geq s_1$, $H^1(Y_d, \mathfrak{N}_d^{\,s}) = 0$ and there is a canonical embedding of Y_d into $\boldsymbol{P}(H^0(Y_d, \mathfrak{N}_d^{\,s})^*)$. For $s > s_1$, we have $H^1(Y_d, Q_d^{\,s}) = 0$; by the results of [4], p. 15-02 to 15-04, we can conclude that $R^1 f_*(Q^s) = 0$ in a neighborhood of d, and that the map

$$(**) \qquad R^0 f_*(Q^s)_d \otimes ({}_D\mathcal{O}_d/\mathfrak{M}_d) \to H^0(Y_d, Q_d^{\,s})$$

is surjective, where $\mathfrak{M}_d$ is the ideal sheaf of $\{d\}$ in D. The first result truncates the sequence $(*)$.

Using hypotheses (4), (5), and (6), we can apply theorem 1 to show that $R^1 f_*(\mathcal{E}^s)$ and $R^0 f_*(\mathcal{E}^s)$ are coherent for all s (but for X smaller.) Hence the stalk $R^1 f_*(\mathcal{E}^s)_d$ is a finitely generated module over ${}_D\mathcal{O}_d$. For $t > s_1$, consider the increasing sequence $\{K_{2j}\}$ of submodules of $R^1 f_*(\mathcal{E}^t)_d$ defined by

$$K_{2j} = \operatorname{Kernel}\{h_* \circ \cdots \circ h_* : R^1 f_*(\mathcal{E}^t)_d \to R^1 f_*(\mathcal{E}^{t+2j})_d\};$$

as ${}_D\mathcal{O}_d$ is Noetherian, the sequence becomes stationary for j greater than some j_0. Hence for $s > t + 2j_0$, we have $R^1 f_*(\mathcal{E}^s)_d \cong R^1 f_*(\mathcal{E}^{s+2})_d$. Thus for s large we can combine $(*)$ and $(**)$ to get

$$(***) \qquad R^0 f_*(\mathcal{E}^s)_d \to R^0 f_*(Q^s)_d \to H^0(Y_d, \mathfrak{N}_d^{\,s}) \oplus H^0(Y_d, \mathfrak{N}_d^{\,s-1}),$$

where both maps are surjective.

Let $L = R^0 f_*(\mathcal{E}^s)_d \otimes ({}_D\mathcal{O}_d/\mathfrak{M}_d)$ and let L^* be its dual space; L^* is isomorphic to C^{m+1} for some m. For a neighborhood B of d, we get a canonical map

$$g : X \mid B \to \boldsymbol{P}^m \times B,$$

where $g(x)$ is the linear functional defined by evaluation at x. We have a canonical section h^s of $\mathcal{E}^s$; we will take the dual of h^s as the first coordinate of L^*.

LEMMA. *The map g is non-singular in a neighborhood of Y_d.*

Proof. As in [6], Theorem 3, we have guaranteed by (***) that we can extend the elements in $H^0(Y_d, \mathfrak{N}_d^s) \oplus H^0(Y_d, \mathfrak{N}_d^{s-1})$ to give a map which is nonsingular at each point of Y_d, hence in a neighborhood of Y_d.

If we define H the hyperplane at ∞ to be the points where the first homogeneous coordinate vanishes, then $g^{-1}(H) = Y$, as $h^s(x) = 0$ only on Y. This concludes the proof.

COROLLARY. *Assumptions* (5) *and* (6) *follow from the other assumptions.*

Proof. We can apply Theorem 3 of [6] to the normal bundle $N_d \to Y_d$ to obtain a neighborhood L_d of Y_d in N_d and a proper injection

$$F:(L_d - Y_d) \to C^k - \Delta(O, R),$$

for $\Delta(O, R)$ the polydisk of radius R centered at 0. Let $\varphi = \Sigma z_i \bar{z}_i$ on C^k; then $\varphi \circ F$ is a strictly plurisubharmonic function on $L_d - Y_d$. By Satz 6, p. 350 of [3], we can extend $\varphi \circ F$ to a strictly plurisubharmonic function φ_0 on a neighborhood L of $L_d - Y_d$ in E. Let $r_*, r_\#$ be real numbers such that $kR^2 < r_* < r_\# < \infty$; then for a sufficiently small neighborhood B of d, the set $f^{-1}(B) \cap \{r_* < \varphi_0 < r_\#\}$ has compact closure in L. Extend φ_0 to be defined on a neighborhood (again called L) of $Y \mid B$ in $E \mid B$, with $\varphi_0 \geq r_\#$ on the extended part. Shrink B so that the set

$$\partial\{x \in L:\varphi_0(x) > r_*\} \cap f^{-1}(B) \quad \text{is contained in} \quad \{x \in L:\varphi_0(x) = r_*\}.$$

Choose r such that $r_* < r < r_\#$, and let $U = \{x \in L:\varphi_0(x) > r\}$. Choose a relatively compact set $B' \subset B$ and a neighborhood W of Y in X. For $r' \in \mathbf{R}$ sufficiently large, we have

$$r' \cdot h(\partial W \cap f^{-1}(B')) \cap U = \varnothing.$$

Let $W_r = \{x \in W \mid B' : r' \cdot h(x) \in U\}$. Then $f: W_r \to B'$ is 1-pseudoconcave, with exhaustion function $\varphi_0 \circ (r' \cdot h)$ and concavity bounds $(r, r_\#)$.

2. Extension to a subvariety

PROPOSITION 2. *Let f, X, Y, D be as in the main theorem, and assume we have the map $g:X \to \mathbf{P}^m \times D$ constructed in Proposition* 1 *(for a restricted family). Then for $d \in D$ there is a neighborhood B of d, a neighborhood U of $Y \mid B$, and a subvariety V of $\mathbf{P}^m \times B$ such that $g(U) \subset V$ and $V_{d'}$ is of pure dimension k for each $d' \in B$.*

Proof. Let $g(X) = W$; consider $W_d - g(Y_d)$ to be in $\mathbf{C}^m$. By theorem 6, p. 229 of [5], there is a closed subvariety V_d' of $\mathbf{C}^m$ such that, for R large enough,

$$V_d' \cap (\mathbf{C}^m - \Delta(0, R)) = (W_d - g(Y_d)) \cap (\mathbf{C}^m - \Delta(0, R)).$$

If we attach $g(Y_d)$, we get a projective variety V_d of pure dimension k.

Let F be an $(m - k - 1)$-dimensional plane in $H \times \{d\}$ such that $F \cap V_d =$

$\emptyset$. We can assume $F = \{(0, \cdots, 0, z_{k+1}, \cdots, z_m)\}$. Define the projection map

$$\pi : (P^m - F) \times D \to P^k \times D$$

by projection on the first $(k + 1)$ homogeneous coordinates.

LEMMA. *$\pi : V_d \to P^k \times \{d\}$ is light.*

Proof. Let $z \in P^k \times \{d\}$, and let $v \in \pi^{-1}(z)$. Define $F \vee v$ as $\{w + v : w \in F\}$ where we add in homogeneous coordinates. Then $v \notin F$ implies $F \vee v$ is $(m - k)$-dimensional.

Assume $V_d \cap (F \vee v)$ has dimension 1, i.e. π is not light. Then as F is of co-dimension 1 in $F \vee v$, we know $F \cap V_d$ has dimension 0, hence is non-empty, which contradicts our choice of F.

As V_d is compact of pure dimension k, we also have that π is proper and surjective.

Let B be a neighborhood of d such that $g(Y_{d'}) \cap (F \times \{d'\}) = \emptyset$ for all d' in B. Let $W' = W - g(Y)$. Then if R is sufficiently large and B is a little smaller, the map

$$\pi : \pi^{-1}(C^k - \overline{\Delta(0, R)}) \cap W' \mid B \to (C^k - \overline{\Delta(0, R)}) \times B$$

is light, proper, and surjective.

For B' a sufficiently small neighborhood of d in B, there is an analytic cover $\sigma : B' \to \Delta$, where Δ is a polydisk in some C^n; as σ is light, proper, and surjective, so is the map $(\mathrm{id} \times \sigma) \circ \pi$. We can apply Theorem 2 to obtain a subvariety V' of C^m which extends $\pi^{-1}(C^k - \overline{\Delta(0, R)}) \cap W' \mid B'$.

We can attach $g(Y \mid B')$ to V' to obtain a subvariety V of $P^m \times B'$; all we need to know is that V is closed, i.e., that, in the notation of [5], p. 227, $\partial W' \mid B' = \partial V'$. But V' is connected at $\partial V'$ (Theorem 2, p. 227 of [5]), and is bounded on $\pi^{-1}(\partial\Delta(0, R + 1))$. Hence the part we have added is bounded, and V is closed.

3. Patching

In this section we will show that the subvarieties we obtain are essentially unique, hence that we can patch together the local information to get a global family of projective varieties. We first investigate the way our embeddings vary with respect to the power of $\mathcal{E}^s$.

Our sheaf sequence

$$0 \to \mathcal{I} \to \mathcal{O} \to \mathcal{O}/\mathcal{I} \to 0$$

is induced by the map $h : \mathcal{I} \to \mathcal{O}$, where h is multiplication by the canonical section h of $\mathcal{E}$. We get an induced surjective map

$$h^* : (L^{s+1})^* \to (L^s)^*.$$

We want to show h^* is compatible with the injections g^{s+1} and g^s. With respect to suitable coordinates, h^* is projection followed by multiplication by h; further, the first coordinates are the special ones previously defined. If K = kernel (h^*), then the induced map $h^p: \boldsymbol{P}((L^{s+1})^* - K) \to \boldsymbol{P}((L^s)^*)$ is the same as projection using homogeneous coordinates. Note that $g^{s+1}(X - Y) \cap K = \varnothing$, as h^{s+1} is non-zero on $X - Y$. Hence $h^p \circ g^{s+1} = g^s$ on $X - Y$. As K is contained in the hyperplane at ∞ of $\boldsymbol{P}((L^{s+1})^*)$, we get a projection map $h^{\#}: \boldsymbol{C}^{m'} \to \boldsymbol{C}^m$ of the affine spaces. We can fill in $g^{s+1}(X - Y)$ and $g^s(X - Y)$ to obtain subvarieties V^{s+1} of $\boldsymbol{C}^{m'}$ and V^s of $\boldsymbol{C}^m$.

LEMMA. *$h^{\#}: V^{s+1} \to V^s$ is an isomorphism.*

Proof. For a sufficiently large polydisk $\Delta \subset \boldsymbol{C}^{m'}$, we have that $h^{\#}(V^{s+1} \cap (\boldsymbol{C}^{m'} - \Delta))$ is a closed subset of $\boldsymbol{C}^m$ contained in V^s. As $h^{\#}$ is a closed map on the compact set $\bar{\Delta}$, the set $h^{\#}(V^{s+1} \cap \bar{\Delta})$ is closed; thus $h^{\#}(V^{s+1})$ is a closed subvariety of $\boldsymbol{C}^m$ which extends $g^s(X - Y)$. The proof of theorem 2 shows that the extension of $g^s(X - Y)$ is unique; hence $h^{\#}: V^{s+1} \to V^s$ is an isomorphism.

To complete the main theorem we cover D by a locally finite open cover $\{B_i\}$ such that over each B_i we have an injection $g_i: W_i \to V_i$. We take the disjoint union V of the varieties, modulo the isomorphisms in the intersections $B_i \cap B_j$. As the cover is locally finite, the embedding $g: W \to V$ is defined for some neighborhood W of Y.

Remark: If D is compact, we can take a finite value of s that will work over all of D; if further $R^0 f_*(\mathcal{E}^s)$ is locally free (for instance, if $R^1 f_*(\mathcal{E}^s) = 0$), then V is a subvariety of the family $f_p: \boldsymbol{P}(R^0 f_*(\mathcal{E}^s)^*) \to D$ of projective spaces.

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REFERENCES

[1] ANDREOTTI, A., and H. GRAUERT, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90**, 193–259 (1962).
[2] GRAUERT, H. "Ein theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen", Inst. Hautes Études Sci. Publ. Math. **5**, 1960.
[3] GRAUERT, H., *Uber Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146**, 331–68 (1962).
[4] GROTHENDIECK, A., Techniques de Construction en Géométrie Analytique, Séminaire Henri Cartan, 13e Année, #15 (1960–1961).
[5] GUNNING, R. C., and H. ROSSI, Analytic functions of several complex variables, Prentice-Hall, 1965.
[6] ROSSI, H., Attaching Analytic Spaces to an Analytic Space Along a Pseudoconcave Boundary, Proceedings of the Conference on Complex Analysis, Minneapolis, 1964, 242–256, Springer-Verlag (1965).
[7] ROSSI, H., *Continuation of Subvarieties of Projective Varieties*, Amer. J. Math., **XCI**, #2, 565–75 (April, 1969).
[8] SIU, Y. T., "A pseudoconcave Generalization of Grauert's Direct Image Theorem". Ann. Scuola Norm. Sup. Pisa **24**, 279–330 and 439–89, (1970).