SOME LIMIT THEOREMS IN RENEWAL THEORY

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In the author's work [5], which contains a section (§4) of renewal theoretic results that are technical lemmata for that work, for brevity only a few of those results were proven. This paper contains some of the proofs omitted from [5] whose availability in connection with it has been suggested, and further related results. The order of presentation is determined by that of [5], and the technicalities involved in some of the statements, and some of the statements themselves, become relevant in the context of [5]; other results have their own interest, such as the generalization of the elementary renewal theorem (Corollary 4); the limit moments of the interarrival time containing the present epoch, the spent waiting time and the residual waiting time (Proposition 5); and the limit joint conditional distribution, given the present epoch, of interarrival times attached and prior to a future epoch (Proposition 9).

All the statements refer to a sequence (the interarrival times) $\tau_1, \tau_2, \cdots$ of nonnegative, nonidentically zero, independent random variables with common distribution μ, and to the random variable (the number of renewal epochs up to the time t) $N(t) = \max \{n : S_n \leq t\}$, $t \geq 0$, where $S_0 = 0$ and $S_n = \sum_{i=1}^{n} \tau_i$, $n \geq 1$. We define $\tau_i = 0$ for $i \leq 0$. The underlying probability space is denoted $(\Omega, \mathfrak{F}, P)$.

The following notation is used. The symbols $\mathfrak{D}$, P and *a.s.* are abbreviations, respectively, for "in distribution", "in probability" and "almost surely". E denotes expectation, $E[\ |\mathfrak{B}]$ conditional expectation with respect to the Borel field $\mathfrak{B}$, and $\mathfrak{F}\{X_\alpha, \alpha \in \mathfrak{A}\}$ the Borel field generated by the random elements X_α, $\alpha \in \mathfrak{A}$. The distribution of the random element X is denoted μ_X. [ ] stands for the integral part of a real number, and $^+$ for the positive part. 1_S is the indicator function of the set S, and R^n is n-dimensional Euclidean space.

Proposition 1. *If $E\tau_1^\alpha < \infty$ for given $\alpha > 0$, then*

$$\frac{1}{t} \max\nolimits_{1 \leq i \leq N(t^\alpha)+1} \tau_i \xrightarrow{P} 0 \quad as \quad t \to \infty.$$

Proof. Since $N(t)/t \xrightarrow{a.s.} 1/E\tau_1$ as $t \to \infty$ ([3], p. 127), then for arbitrary $\epsilon > 0$ and $\delta > 0$, writing $m = 1/E\tau_1$ we have

$$\begin{aligned}
&\limsup\nolimits_{t\to\infty} P[\max\nolimits_{1\leq i \leq N(t^\alpha)+1} \tau_i \geq t\epsilon] \\
&\leq \limsup\nolimits_{t\to\infty} P\left[\max\nolimits_{1\leq i\leq N(t^\alpha)+1} \tau_i \geq t\epsilon, \frac{N(t^\alpha)}{t^\alpha} < m + \delta\right] \\
&\leq \limsup\nolimits_{t\to\infty} P[\max\nolimits_{1\leq i\leq (m+\delta)t^\alpha+1} \tau_i \geq t\epsilon] \\
&= \limsup\nolimits_{t\to\infty} (1 - P[\tau_1 < t\epsilon]^{[(m+\delta)t^\alpha]+1}) \\
&\leq 1 - \liminf\nolimits_{t\to\infty} \left(1 - \frac{(t\epsilon)^\alpha P[\tau_1 \geq t\epsilon]}{(t\epsilon)^\alpha}\right)^{(m+\delta)t^\alpha} P[\tau_1 < t\epsilon] = 0,
\end{aligned}$$

because $E\tau_1^\alpha < \infty$ implies that $t^\alpha P[\tau_1 \geq t] \to 0$ as $t \to \infty$.

Remark. This result is valid even if $E\tau_1 = \infty$ (when $\alpha < 1$).

COROLLARY 1. *If* $E\tau_1^{\alpha} < \infty$ *for given* $\alpha > 0$, *then*

$$\frac{1}{t}\,\tau_{N(t^{\alpha})+1} \xrightarrow{P} 0 \quad \text{and} \quad \frac{1}{t}\,\sup_{0\le s\le t^{\alpha}}(s - S_{N(s)}) \xrightarrow{P} 0$$

as $t \to \infty$.

Proof. The second assertion follows from

$$\sup_{0\le s\le t}\,(s - S_{N(s)}) \le \max_{1\le i\le N(t)+1} \tau_i\,.$$

The second statement of this corollary is Lemma 4.1 of [5].

Let η, θ_0, θ_1, $\cdots$ be a sequence of random elements (of some topological space), with θ_0 constant, which is independent of the sequence τ_1, τ_2, $\cdots$, and consider the Borel fields $\mathcal{G}_n = \mathcal{F}\{\eta, \theta_0, \cdots, \theta_{n-1}, \tau_1, \cdots, \tau_{n+1}\}$ for each $n \ge 1$, $\mathcal{G}_0 = \mathcal{F}\{\tau_1\}$, and $\mathcal{G}_{-1} = \{\varnothing, \Omega\}$, and the Borel field $\mathcal{F}_t$ generated by η, $\theta_0, \cdots, \theta_{n-1}, \tau_1, \cdots, \tau_{n+1}$ up to the random index $N(t)$, that is

$$\mathcal{F}_t = \{A \in \mathcal{F}: A \cap [N(t) = n] \in \mathcal{G}_n \text{ for each } n \ge 0\},$$

for each $t > 0$.

PROPOSITION 2. *For any random vector* X *and for each* $n \ge 0$ *and* $t > 0$

$$E[1_{[N(t)=n]}X \mid \mathcal{F}_t] \overset{a.s.}{=} E[1_{[N(t)=n]}X \mid \mathcal{G}_n].$$

Proof. Let $S = [N(t) = n]$. For $A \in \mathcal{F}_t$,

$$\int_A 1_S X\,dP = \int_{A\cap S} X\,dP = \int_{A\cap S} E[X \mid \mathcal{G}_n]\,dP = \int_A 1_S E[X \mid \mathcal{G}_n]\,dP$$
$$= \int_A E[1_S X \mid \mathcal{G}_n]\,dP,$$

and the conclusion follows from the fact that $E[1_{[N(t)=n]}X \mid \mathcal{G}_n]$ is measurable with respect to $\mathcal{F}_t$.

This is Lemma 4.2 of [5].

PROPOSITION 3. *Let* $g_1, \cdots, g_k$ *be the elementary symmetric functions of* k *arguments (i.e., if* $x_1, \cdots, x_k$ *are the arguments,* $\Sigma_i x_i$, $\Sigma_{i<j} x_i x_j$, $\cdots$, $\Pi_i x_i$) *and let* $i_1, \cdots, i_j$ *be integers such that* $1 \le i_1 < \cdots < i_j \le k$. *For any measurable vector* X *defined on* R^{∞} *or* R^m *for appropriate* m, *and for each* $n \ge 0$ *and* $k \ge 1$ *and any permutation* σ *of* $\{1, \cdots, k\}$ *we have, with the notation* $G_1 = g_1(\tau_{n+1}, \cdots, \tau_{n+k})$ $G_k = g_k(\tau_{n+1}, \cdots, \tau_{n+k})$,

$$E[X(\tau_1, \cdots, \tau_n; \tau_{n+i_1}, \cdots, \tau_{n+i_j}, G_1, \cdots G_k, \tau_{n+k+1}, \cdots) \mid \mathcal{G}_{n-1}]$$
$$\overset{a.s.}{=} E[X(\tau_1, \cdots, \tau_n, \tau_{n+\sigma(i_1)}, \cdots, \tau_{n+\sigma(i_j)}, G_1, \cdots, G_k, \tau_{n+k+1}, \cdots) \mid \mathcal{G}_{n-1}].$$

Proof. For $A \in \mathcal{G}_{n-1}$,

$$\int_A X(\tau_1, \cdots, \tau_n, \tau_{n+i_1}, \cdots, \tau_{n+i_j}, G_1, \cdots, G_k, \tau_{n+k+1}, \cdots)\,dP$$
$$= \int_{R^n} E[1_A X(\ ) \mid \tau_1 = x_1, \cdots, \tau_n = x_n]\mu(dx_1) \cdots \mu(dx_n)$$
$$= \int_{R^n} h(x_1, \cdots, x_n) EX(x_1, \cdots, x_n, \tau_{n+i_1}, \cdots, \tau_{n+i_j}, G_1, \cdots, G_k, \tau_{n+k+1}, \cdots)\mu(dx_1) \cdots \mu(dx_n),$$

where $h(x_1, \cdots, x_n) = E[1_A \mid \tau_1 = x_1, \cdots, \tau_n = x_n]$; hence it suffices to show that

$$(\tau_{i_1}, \cdots, \tau_{i_j}, G_1, \cdots, G_k) \overset{\mathfrak{D}}{=} (\tau_{\sigma(i_1)}, \cdots, \tau_{\sigma(i_j)}, G_1, \cdots, G_k),$$

but this is obvious.

COROLLARY 2. *For each $n \geq 0$ and $k \geq 1$, for any permutation σ of $\{1, \cdots, k\}$ and any integers $i_1, \cdots, i_j$ such that $1 \leq i_1 < \cdots < i_j \leq k$, and for any measurable vector X defined on R^∞ or R^m for appropriate m*

$$E[1_{[N(t)=n+k]}X(\tau_1, \cdots, \tau_n, \tau_{n+i_1}, \cdots, \tau_{n+i_j}, \tau_{n+k+1}, \cdots) \mid \mathcal{G}_{n-1}]$$

$$\overset{a.s.}{=} E[1_{[N(t)=n+k]}X(\tau_1, \cdots, \tau_n, \tau_{n+\sigma(i_1)}, \cdots, \tau_{n+\sigma(i_j)}, \tau_{n+k+1}, \cdots) \mid \mathcal{G}_{n-1}],$$

and in particular (with $n = 0$)

$$\int_{[N(t)=k]} X(\tau_{i_1}, \cdots, \tau_{i_j}, \tau_{k+1}, \cdots)\, dP = \int_{[N(t)=k]} X(\tau_{\sigma(i_1)}, \cdots, \tau_{\sigma(i_j)}, \tau_{k+1}, \cdots)\, dP.$$

Proof. $N(t) = n + k$ if and only if

$$\sum_{i=1}^{n} \tau_i + \sum_{i=1}^{k} \tau_{n+i} \leq t < \sum_{i=1}^{n} \tau_i + \sum_{i=1}^{k} \tau_{n+i} + \tau_{n+k+1}.$$

Apply Proposition 3 with $1_{[N(t)=n+k]}X$ in place of X.

This result is Lemma 4.3 of [5].

As a simple application of Corollary 2, Wald's equation

$$ES_{N(t)+1} = E\tau_1(EN(t) + 1),$$

and Proposition 5, we obtain

$$\lim_{t\to\infty} \operatorname{Cov}(\tau_1, N(t)) = -\frac{\operatorname{Var} \tau_1}{E\tau_1}.$$

PROPOSITION 4. *For all $\alpha > 0$ and all integers n and m such that $0 \leq m \leq n$,*

$$E\tau^\alpha_{N(t)-n} \leq E\tau^\alpha_{N(t)-m} \leq E\tau^\alpha_{N(t)+1}.$$

Proof. First we see that $E\tau^\alpha_{N(t)-n} \leq E\tau^\alpha_{N(t)-m}$, for by Corollary 2 we have for all $a > 0$

$$\begin{aligned} P[\tau_{N(t)-n} \geq a] &= \sum_{k=n}^{\infty} P[\tau_{k-n} \geq a, N(t) = k] \\ &= \sum_{k=n}^{\infty} P[\tau_k \geq a, N(t) = k] \leq \sum_{k=m}^{\infty} P[\tau_k \geq a, N(t) = k] \\ &= \sum_{k=m}^{\infty} P[\tau_{k-m} \geq a, N(t) = k] = P[\tau_{N(t)-m} \geq a]. \end{aligned}$$

To prove that $E\tau^\alpha_{N(t)} \leq E\tau^\alpha_{N(t)+1}$ it suffices to show that $P[\tau_{N(t)+1} \geq a] \geq$

$P[\tau_{N(t)} \geq a]$ for all $a > 0$, which is true if $P[\tau_{k+1} \geq a, N(t) = k] \geq P[\tau_k \geq a, N(t) = k]$ for all $a > 0$ and all k, which is obvious for $a > t$; for $a \leq t$

$$P[\tau_{k+1} \geq a, N(t) = k]$$
$$= \int_0^t \mu(dx_1) \int_0^{t-x_1} \mu(dx_2) \cdots \int_0^{t-x_1-\cdots-x_{k-1}} \mu(dx_k) \int_{\max\{a, t-x_1-\cdots-x_k\}}^{\infty} \mu(dx_{k+1})$$

and

$$P[\tau_k \geq a, N(t) = k]$$
$$= \int_0^t \mu(dx_1) \int_0^{t-x_1} \mu(dx_2) \cdots \int_a^{t-x_1-\cdots-x_{k-1}} \mu(dx_k) \int_{t-x_1-\cdots-x_k}^{\infty} \mu(dx_{k+1});$$

hence it is enough to show that

$$\int_0^x \mu(dx_1) \int_{\max\{a, x-x_1\}}^{\infty} \mu(dx_2) \geq \int_a^x \mu(dx_1) \int_{x-x_1}^{\infty} \mu(dx_2),$$

which is obvious for $x < a$; for $x \geq a$ this is done by straightforward calculation, or by looking at the regions of integration, using Fubini's theorem (consider the cases $a \geq x/2$ and $a < x/2$ separately).

Proposition 5. *If* $E\tau_1^{\alpha+1} < \infty$ *for given* $\alpha > 0$, *then in both the lattice and the nonlattice cases* ([2], p. 54)

$$\lim_{t\to\infty} E\tau_{N(t)+1}^{\alpha} = \frac{E\tau_1^{\alpha+1}}{E\tau_1},$$

and in the nonlattice case

$$\lim_{t\to\infty} E(t - S_{N(t)})^{\alpha} = \lim_{t\to\infty} E(S_{N(t)+1} - t)^{\alpha} = \frac{E\tau_1^{\alpha+1}}{(\alpha + 1)E\tau_1}.$$

Proof. Let λ denote the distribution of $\tau_{N(t)+1}$, and ν the renewal measure defined by $\nu((x_1, x_2]) = E(N(x_2) - N(x_1))$ with an atom of unit weight at the origin. Then ([4], p. 371)

$$\lambda([0, x]) = \int_{t-x}^{t} \mu((t - y, x])\nu(dy).$$

It is easy to verify that λ is absolutely continuous with respect to μ and that the Radon-Nikodým derivative of λ with respect to μ is $d\lambda/d\mu(x) = \nu((t - x, t])$. Hence

$$E\tau_{N(t)+1}^{\alpha} = \int_0^{\infty} x^{\alpha}\nu((t - x, t])\mu(dx).$$

Since $\nu((t - x, t]) \leq K + Lx$ for all t and x, where K and L are some constants (see [4], p. 360, or Propositions 7 and 6), the first conclusion is obtained from the Blackwell renewal theorem in both the lattice and the nonlattice cases ([2], p. 219) and the dominated convergence theorem.

In the nonlattice case ([4], p. 369–371)

$$t - S_{N(t)} \xrightarrow{\mathfrak{D}} \theta \quad \text{and} \quad S_{N(t)+1} - t \xrightarrow{\mathfrak{D}} \theta \quad \text{as } t \to \infty,$$

where θ is the distribution defined by

$$\theta([0, x]) = \frac{1}{E\tau_1} \int_0^x P[\tau_1 > s]\, ds.$$

Since $t - S_{N(t)} \leq \tau_{N(t)+1}$ and $S_{N(t)+1} - t \leq \tau_{N(t)+1}$, and $\{\tau_{N(t)+1}^{\alpha}, t \geq 0\}$ is uniformly integrable, then ([1], p. 32)

$$\lim_{t\to\infty} E(t - S_{N(t)})^{\alpha} = \lim_{t\to\infty} E(S_{N(t)+1} - t)^{\alpha}$$

$$= \int_0^\infty x^{\alpha}\theta(dx) = \frac{1}{E\tau_1} \int_0^\infty x^{\alpha} P[\tau_1 > x]\, dx$$

$$= \frac{1}{(\alpha + 1)E\tau_1} \int_0^\infty P[\tau_1^{\alpha+1} > x^{\alpha+1}]\, dx^{\alpha+1} = \frac{E\tau_1^{\alpha+1}}{(\alpha + 1)E\tau_1}.$$

COROLLARY 3. *If $E\tau_1^{\alpha+1} < \infty$ for given $\alpha > 0$, then for every integer $n \geq 0$*

$$\sup_t E\tau_{N(t)+1-n}^{\alpha} < \infty.$$

Proof. It follows from Proposition 4 and the proof of Proposition 5.

This result is Lemma 4.4 of [5].

From Proposition 5 the following limit covariances are easily obtained for the nonlattice case:

$$\lim_{t\to\infty} \text{Cov}\,(t - S_{N(t)}, \tau_{N(t)+1}) = \lim_{t\to\infty} \text{Cov}\,(S_{N(t)+1} - t, \tau_{N(t)+1})$$

$$= \frac{E\tau_1^3 E\tau_1 - (E\tau_1^2)^2}{2(E\tau_1)^2},$$

$$\lim_{t\to\infty} \text{Cov}\,(t - S_{N(t)}, S_{N(t)+1} - t) = \frac{2E\tau_1^3 E\tau_1 - 3(E\tau_1^2)^2}{12(E\tau_1)^2}.$$

PROPOSITION 6. *For every integer $m \geq 1$ there is a constant K such that for all $t \geq 0$*

$$EN(t)^m \leq K \max\{1, t^m\}.$$

Proof. (See [3], p. 127) Since τ_1 is not identically zero there is a $\delta > 0$ such that $P[\tau_1 \geq \delta] = p > 0$. Let $\tau_i' = \delta$ if $\tau_i \geq \delta$ and $\tau_i' = 0$ if $\tau_i < \delta$, and let $N'(t) = \max\{k : \Sigma_{k=1}^{i} \tau_i' \leq t\}$, $t \geq 0$. Clearly $N'(t) \geq N(t)$, and hence

$$EN'(t)^m \geq EN(t)^m,$$

so that it suffices to prove the lemma for $N'(t)$. $N'(t) - [t/\delta]$ has the negative binomial distribution with parameters $[t/\delta] + 1$ and p, and in particular

$$P[N'(\delta k) = n] = \binom{n}{k} p^{k+1}(1 - p)^{n-k}, \qquad n \geq k,$$

where k is an integer. Using the fact that $N'(t) \geq n$ if and only if $\Sigma_{i=1}^{n}\tau_i' \leq t$ one easily obtains

$$EN'(t)^m \leq \sum_{k\leq(t/\delta)} \frac{1}{p} \sum_{n\geq 1} \binom{n}{k} p^{k+1}(1-p)^{n-k}f(n),$$

where $f(n)$ is the number of integers j such that $n^m \leq j < (n+1)^m$. Now

$$f(n) \leq 1 \Big/ \frac{d}{dx} x^{1/m} \Big|_{x=(n+1)^m} = m(n+1)^{m-1},$$

and hence

$$EN'(t)^m \leq \sum_{k\leq(t/\delta)} \frac{m}{p} E(N'(\delta k) + 1)^{m-1},$$

and since $N'(\delta k) \leq N'(t)$ for $k \leq t/\delta$,

$$EN'(t)^m \leq \frac{1}{p}\left(\left[\frac{t}{\delta}\right] + 1\right) mE(N'(t) + 1)^{m-1}.$$

This expression is valid for every integer $m \geq 1$, and therefore the conclusion follows from it by induction on m and the fact that $EN'(t)^m$ increases with t.

This proposition is Lemma 4.5 of [5].

The following result is used in the proof of Lemma 4.6 of [5].

PROPOSITION 7. *Let T and H be nonnegative random variables that are jointly independent of $\{\tau_i, i \geq 1\}$ and such that $T \leq H$ a.s. Then for every integer $m \geq 0$*

$$E(N(H) - N(T))^m \leq E(1 + N(H - T))^m.$$

Proof. We will first show that for all $k \geq 0$

$$E[(N(H) - N(T) - 1)^+]^k \leq EN(H - T)^k.$$

For each $a > 0$, denoting (a) = smallest integer $\geq a$,

$$\begin{aligned} P[N(H) - N(T) - 1 \geq a] &= \sum_{r=0}^{\infty} P[N(H) \geq r + 1 + (a), S_r \leq T < S_{r+1}] \\ &= \sum_{r=0}^{\infty} P[S_{r+1+(a)} \leq H, S_r \leq T < S_{r+1}] \\ &\leq \sum_{r=0}^{\infty} P[\textstyle\sum_{i=r+2}^{r+1+(a)} \tau_i \leq H - T, N(T) = r]; \end{aligned}$$

using the independence hypothesis,

$$\begin{aligned} P[\textstyle\sum_{i=r+2}^{r+1+(a)} \tau_i \leq H - T, N(T) = r] &= \int P[\textstyle\sum_{i=r+2}^{r+1+(a)} \tau_i \leq H - t, N(t) = r \mid T = t]\mu_T(dt) \\ &= \int P[S_{(a)} \leq H - t \mid T = t]P[N(t) = r]\mu_T(dt); \end{aligned}$$

hence, by the monotone convergence theorem,

$$P[N(H) - N(T) - 1 \geq a] \leq \int P[S_{(a)} \leq H - t \mid T = t]\mu_T(dt)$$
$$= P[S_{(a)} \leq H - T] = P[N(H - T) \geq a],$$

which implies the desired result.

Now,

$$E(N(H) - N(T))^m \leq E[(N(H) - N(T) - 1)^+ + 1]^m$$
$$= \textstyle\sum_{k=0}^{m} \binom{m}{k} E[(N(H) - N(T) - 1)^+]^k,$$

which together with the above inequality yields the conclusion.

The next proposition is the first assertion of Lemma 4.6 of [5], where the proof of the lemma appears.

Proposition 8. *Let t and h be real numbers such that $0 \leq t \leq h$, T_n and H_n nonnegative random variables that are jointly independent of $\{\tau_i, i \geq 1\}$ and such that $T_n \leq H_n$ a.s., $T_n \xrightarrow{a.s.} t$ and $H_n \xrightarrow{a.s.} h$ as $n \to \infty$, and for a positive integer m and some $\delta > 0$, $\sup_n EH_n^{m+\delta} < \infty$, and let $b_1, b_2, \cdots$ be a sequence of positive real numbers converging to infinity. Then*

$$\lim_{n\to\infty} E\left|\frac{N(b_n H_n) - N(b_n T_n)}{b_n} - \frac{h - t}{E\tau_1}\right|^m = 0.$$

An immediate consequence of this proposition is the following generalization of the elementary renewal theorem.

Corollary 4. *For any real numbers α and β such that $0 \leq \alpha < \beta$, and any $\delta > 0$,*

$$\lim_{t\to\infty} E\left(\frac{N(\beta t) - N(\alpha t)}{t}\right)^\delta = \left(\frac{\beta - \alpha}{E\tau_1}\right)^\delta.$$

The following result is contained in the proof of Lemma 4.7 of [5].

Proposition 9. *If $E\tau_1 < \infty$, if for $p > 1$ a measurable function $f: R^{m+1} \to R$ satisfies $E|f(\tau_1, \cdots, \tau_{m+1})|^p < \infty$ and $\sup_t E|f(\tau_{N(t)-m}, \cdots, \tau_{N(t)})|^p < \infty$, and if α_t and β_t, $t \geq 0$, are real numbers such that $\liminf_{t\to\infty} \alpha_t > 0$, $\limsup_{t\to\infty} \beta_t < \infty$ and $\liminf_{t\to\infty} (\beta_t - \alpha_t) > 0$, then*

$$\lim_{t\to\infty} E\,|\,E[f(\tau_{N(\beta_t t)-m}, \cdots, \tau_{N(\beta_t t)}) \mid \mathfrak{F}_{\alpha_t t}] - Ef(\tau_1, \cdots, \tau_{m+1})\,| = 0$$

($\mathfrak{F}_{\alpha_t t}$ is as defined above Proposition 2).

Since this statement holds for all bounded continuous functions f, it can be interpreted, in a loose way, by saying that if the difference between a future epoch and the present epoch tends to infinity about linearly, then, conditioned to present epoch, interarrival times attached and prior to the future epoch jointly converge weakly to the product distribution of the measure μ with itself. It follows in particular from Proposition 9 that

$$(\tau_{N(t)-m}, \cdots, \tau_{N(t)}) \xrightarrow{\mathfrak{D}} (\tau_1, \cdots, \tau_{m+1}) \text{ as } t \to \infty,$$

which implies that the highly dependent random variables $\tau_{N(t)-m}, \cdots, \tau_{N(t)}$ are asymptotically independent as $t \to \infty$.

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