NOTE ON THE GENERALIZED VECTOR FIELD PROBLEM

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1. Introduction

Let ξ_k, η_k, and γ_k denote the Hopf real, complex, and quaternionic line bundles over RP^k, CP^k, and QP^k respectively. To determine the real geometric dimensions of $m\xi_k$, $m\eta_k$, and $m\gamma_k$ for $m > k$ is the generalized vector field problem. In this note we apply complex K-theory following the procedure of [1] to produce elementary proofs for the following theorems.

THEOREM A. *Suppose $2m\eta_k$ has real geometric dimension $\leqq 2t + \epsilon$ where $\epsilon = 0$ or 1. Then $c_{t+j}(2m\eta_k)$ is divisible by 2^j.*

THEOREM B. *Suppose $m\gamma_k$ has real geometric dimension $\leqq 4t + \epsilon$ or $4t + 2 + \epsilon$ where $\epsilon = 0$ or 1. Then $e_{t+j}(m\gamma_k)$ is divisible by 2^{2j} in the first case and by 2^{2j-1} in the second case.*

A strong generalization of Theorem A and a weaker version of Theorem B were communicated to me by S. Gitler and M. Mahowald.

2. Preliminaries

Let KU denote the complex K-theory functor. Let ξ^c denote the complexification of a vector bundle ξ. Then $\eta_k^c = \eta_k \oplus \bar{\eta}_k$ where $\eta_k \otimes \bar{\eta}_k = 1$ and $\gamma_k^c = 2\gamma_k$. $KU(CP^k) = Z[y]/(y^{k+1})$ and $KU(QP^k) = Z[z]/(z^{k+1})$ where $y = \eta_k - 1$ and $z = \gamma_k - 2$ from [3]. Note in $\tilde{K}U(CP^k)$ that

$$(2.1) \qquad \bar{y} = \bar{\eta}_k - 1 = \frac{-y}{1+y}.$$

Let λ^i denote the extension of the i-th exterior power on complex vector bundles to KU. For arbitrary x in $KU(X)$ we define $\lambda_t(x) = \sum_{i=0}^{\infty} \lambda^i(x)t^i$ where t is an indeterminate. Properties of exterior powers imply that

$$(2.2) \qquad \lambda_t(x + y) = \lambda_t(x)\lambda_t(y).$$

Let $R\text{Spin}(n)$ denote the complex representation ring of the spinor group $\text{Spin}(n)$. $R\text{Spin}(2n) = Z[\lambda^1, \cdots, \lambda^{n-2}, \Delta_{2n}^+, \Delta_{2n}^-]$ and $R\text{Spin}(2n+1) = Z[\lambda^1, \cdots, \lambda^{n-1}, \Delta_{2n+1}]$ from [2]. Further,

$$(2.3) \qquad \begin{aligned} i^*\Delta_{2n}^+ = i^*\Delta_{2n}^- &= 2^{n-t-1}(\Delta_{2t}^+ + \Delta_{2t}^-) \\ &= 2^{n-t-1}\Delta_{2t+1} \end{aligned}$$

where $i: \text{Spin}(2t + \epsilon) \to \text{Spin}(2n)$ denotes the standard embedding for $\epsilon = 0$ or 1. We define $\lambda_1 = \Sigma_{i=0}^{2n} \lambda^i$ and $\lambda_{-1} = \Sigma_{i=0}^{2n} (-1)^i\lambda^i$. Then the equalities

$$(2.4) \qquad \lambda_{-1} = (\Delta_{2n}^+ - \Delta_{2n}^-)^2 \text{ and } \lambda_1 - \lambda_{-1} = 4\Delta_{2n}^+\Delta_{2n}^-$$

hold in $R\mathrm{Spin}(2n)$ for even n by [1]. Applying the α construction gives relations in $KU(B\mathrm{Spin}(2n))$ corresponding to (2.4) for n even. Let $C_{r,s}$ denote the binomial coefficient $\binom{r}{s}$.

PROPOSITION 2.5. $\lambda_t(n\eta_k{}^c) = ((1+t)^2 + ty^2/(1+y))^n$ and $\lambda_t(n\gamma_k{}^c) = ((1+t)^2 + tz)^{2n}$.

Proof. $\lambda_t(n\eta_k{}^c) = \lambda_t(n\eta_k \oplus n\bar{\eta}_k) = (\lambda_t(\eta_k)\lambda_t(\bar{\eta}_k))^n = (1 + t(y + \bar{y} + 2) + t^2)^n = ((1+t)^2 + ty^2/(1+y))^n$ by (2.1) and (2.2). Similarly, $\lambda_t(n\gamma_k{}^c) = \lambda_t(2n\gamma_k) = \lambda_t(\gamma_k)^{2n} = (1 + \gamma_k t + t^2)^{2n} = ((1+t)^2 + tz)^{2n}$.

3. Proofs

Proof of A. Set $n = 2m$ and consider the following diagram where $\epsilon = 0$ or 1.

$$
(3.1)\qquad
\begin{array}{ccc}
 & & B\mathrm{Spin}(2t+\epsilon) \\
 & \overset{g}{\nearrow} & \downarrow \pi \\
CP^k & \xrightarrow{\;n\eta_k\;} & B\mathrm{Spin}(2n)
\end{array}
$$

By hypothesis the classifying map for $n\eta_k$ has a lifting g. Let ρ_{2n} denote the universal bundle over $B\mathrm{Spin}(2n)$. Note that $\Delta_{2n}^+(n\eta_k) = \Delta_{2n}^-(n\eta_k)$ since $\pi^*\Delta_{2n}^+(\rho_{2n}) = \pi^*\Delta_{2n}^-(\rho_{2n})$ by (2.3). Thus $[2\Delta_{2n}^+(n\eta_k)]^2 = \lambda_1(n\eta_k{}^c)$ by (2.4) and $\Delta_{2n}^+(n\eta_k)$ is divisible by 2^{n-t-1} by (2.3). Since $\lambda_1(n\eta_k{}^c)$ is not in $\widetilde{KU}(CP^k)$, 2^{2m-t} divides $\sqrt{\lambda_1(n\eta_k{}^c)} = (2+y)^{2m}(1+y)^{-m}$ by (2.5) $= \Sigma_{i=0}^k s_i y^i$ where $s_i = \Sigma_{\ell=0}^i (-1)^{i-\ell}2^{2m-\ell} C_{2m,\ell}C_{m-\ell+i-1,i-\ell}$. Note that 2^{2m-t} divides s_{t+1} so 2 divides $C_{2m,t+1}$. Inductively we assume 2^{j-1} divides $C_{2m,t+j-1}$. Then 2^{2m-t} divides $2^{2m-(t+j)}C_{2m,t+j}$ so that 2^j divides $C_{2m,t+j}$. Since $c_{t+j}(2m\eta_k) = C_{2m,t+j}\beta^{t+j}$ where β generates $\bar{H}^*(CP^k)$, Theorem A is proved.

Proof of B. Assume first that $m\gamma_k$ lifts to $B\mathrm{Spin}(4t+\epsilon)$ for $\epsilon = 0$ or 1. Then $\sqrt{\lambda_1(m\gamma_k{}^c)}$ is divisible by 2^{2m-2t} by the argument for Theorem A. By (2.5) $\sqrt{\lambda_1(m\gamma_k{}^c)} = (z+4)^m = \Sigma_{i=0}^k C_{m,i}2^{2m-2i}z^i$. It follows that $C_{m,t+j}$ and hence the symplectic Pontrjagin class $e_{t+j}(m\gamma_k)$ is divisible by 2^{2j}.

In the second case $m\gamma_k$ lifts to $B\mathrm{Spin}(4t+2+\epsilon)$ with $\epsilon = 0$ or 1. Thus $\Delta_{4m}^+(m\gamma_k)$ is divisible by $2^{2m-2t-2}$ by (2.3) so $\sqrt{\lambda_1(m\gamma_k{}^c)}$ is divisible by $2^{2m-2t-1}$. It follows that $C_{m,t+j}$ is divisible by 2^{2j-1} for $t + j \leq k$ so Theorem B is proved.

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