ON THE GEOMETRIC DIMENSION OF STABLE REAL VECTOR BUNDLES

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1. Let X be a finite t-dimensional CW-complex and let ξ be a real m-plane bundle with $m > t$. We say the geometric dimension of $\xi = g.d.(\xi) \leq t - k$ or codim $(\xi) \geq k$ if there is a $(t-k)$-plane bundle η such that $\xi = \eta \oplus (m - t + k)$ where $\oplus$ is Whitney sum and $(m - t + k) = X \times R^{m-t+k}$. Of course, by the classification theorem for bundles (see [11], 19.3), $g.d.(\xi) \leq t$ and codim $(\xi) \geq 0$.

In [3], Gitler and Mahowald proved the following:

THEOREM *A*. *If* ξ *is trivial over the* $(q-1)$*-skeleton of* X, *then codim* $(\xi) \geq$ *min* $(q - 2\,|\,log_2 t\,| - 5, |\,t/2\,| - 1)$ (*where* $|\,x\,|$ is the greatest integer in x.)

We will prove the following results:

THEOREM 1. *If* $H^\ell(X; Z_2) = 0$, $0 < \ell < q$, *then codim* $(\xi) \geq$ *min* $(q - 2\,|\,log_2 t\,| - 5, |\,t/2\,| - 1)$.

As in [3], a sharper but more complicated result is proven, and this is stated as Theorem 3.5.

In [7], Kervaire proved that if ξ is a bundle over S^{4i}, its i^{th} Pontrjagin class $P_i(\xi) \in H^{4i}; Z)$ is divisible by $d_i(2i-1)!$, where d_i is 1 or 2 as i is even or odd. These numbers will arise later and we will let m_i be the power of 2 in $(2i-1)!$. We also prove:

THEOREM 2. *If* $H^\ell(X; Z)$ *has no 2-torsion for* $\ell < q$ *and* $H^\ell(X; Z_2) = 0$ *for* $\ell \equiv 1, 2\ (8)$ *and* $\ell < q$ *and if* $P_i(\xi)$ *is divisible by* 2^{n_i}, *where* $n_i = m_i + 1 - 2i + |\,(q+1)/2\,|$, *then codim* $(\xi) \geq$ *min* $(q - 2\,|\,\log_2{}^t\,| - 5, |t/2\,| - 1)$.

COROLLARY 3. *If dim* $X = t > 16$ *and* $H^m(X; Z_2) = 0$, $0 < m < t$, *and* ξ *is any bundle over* X (of dim. $> t$), *then codim* $(\xi) \geq [t/2] - 1$.

Using the results of M. Hirsh [5], we get:

COROLLARY 4. *If* M^t *is a differentiable manifold such that* $H^m(M; Z_2) = 0$, $0 < m < q$ *or if* M^t *and its Pontrjagin classes satisfy the hypothesis of Theorem 2, then* M^t *immerses in* R^{2t-x}, *where* $x =$ *min* $(q - 2\,|\,\log_2 t\,| - 5, |t/2\,| - 1)$.

In particular, we get:

COROLLARY 5. *Any lens space* $L_t(p) = S^{2t+1}/Z_p$, *p odd, immerses in* R^{3t+3}.

This result was proven by Uchida in [13], using the fortunate fact that the (stable) tangent bundle of $L_t(p)$ is the restriction of the tangent bundle of $L_{t+1}(p)$ which allowed him to observe that all obstructions were zero.

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To prove our results, we will construct a sequence of spaces E_n which are total spaces of fibration over BSO, the classifying space for stable orientable bundles. We denote by $BSO[m]$ the $(m-1)$-connected covering of BSO, i.e., $f_m: BSO[m] \to BSO$ is a fibration, $\pi_i(BSO[m]) = 0$, $i < m$ and $\pi_i(f_m)$ is an isomorphism for $i \geq m$. (See Hu [6].) Let $P_k \in H^{4k}(BSO; Z)$ be the k^{th} universal Pontrjagin class and let γ_k generate $H^{4k}(BSO[4k]; Z) \cong Z$. Then by [8], $H^*(BSO; Q)$ is a polynomial ring freely generated by (the images under the coefficient homomorphism of) the P_k's, and, as noted above $f_{4k}^*(P_k) = e_k\gamma_k$, where $e_k = \pm d_k(2k-1)!$ and $d_k = 1$ or 2 as k is even or odd.

THEOREM 5. *For* $n = 4a + b$, $0 \leq b \leq 3$, *let* $\varphi(n) = 8a + 2^b$ *and let* $c_n = |[\varphi(n) - 1]/4|$, *where* $|x|$ *is the greatest integer in* x. *There is a sequence of spaces* E_n, $n \geq 1$, *where* $E_1 = BSO$, $E_2 = BSO[4]$ *such that*
(1) *For* $n \geq 2$, $p_n : E_n \to E_{n-1}$ *is a principal fibration with fiber a products of* $K(Z_2, \ )$'s and p_n^* *is* 0 *mod* 2 *in dimensions* $< 2^n$.
(2) *Let* $q_n = p_2 \circ \cdots \circ p_n : E_n \to BSO$. *Then* q_n *is a rational and* Z_p *equivalence, for* p *a prime* $\neq 2$.
(3) *There is a natural map* $i_n: BSO[\varphi(n)] \to E_n$ *so that* $f_{\varphi(m)} = q_n i_n$.
(4) *For* $n \geq 3$, $H^*(E_n; Z_2) \cong \{\otimes_{1 \leq i \leq c_n} H^*(K(Z, 4i); Z_2) \otimes H^*(BSO[\varphi(n)]; Z_2)\}$, *as rings and the factor* $\{\otimes H^*(K(Z, 4i); Z_2)\} \otimes 1$ *is also correct as a subalgebra over the mod 2 Steenrod algebra. For* $1 \leq i \leq c_n$, *denote by* $\beta_{n,i} \in H^{4i}(E_n; Z_2)$ *the generator of* $1 \otimes \cdots \otimes H^{4i}(K(Z, 4i); Z_2) \otimes \cdots \otimes 1$ *and let* $\alpha_n \in H^{\varphi(n)}(E_n; Z_2)$ *denote the generator of* $1 \otimes H^{\varphi(n)}(BO[\varphi(n)]; Z_2)$. *The above splitting is induced in part by* i_n; *in particular,* $i_n^*(\alpha_n)$ *generates* $H^{\varphi(n)}(BO[\varphi(n); Z_2)$. *A map* $F: X \to E_{n-1}$ *will lift to* E_n *if and only if* $F^*(\beta_{n-1,i}) = 0 = F^*(\alpha_{n-1})$, $1 \leq i \leq c_{n-1}$.
(5) *Note that by* (2), $H^*(E_n; Q)$ *is a polynomial algebra. We can choose* $x_{n,k} \in H^{4k}(E_n; Z)$, $k > 0$ *so that*

(a) *When included into* $H^*(\mathrm{E}_n; Q)$, *they freely generate the polynomial algebra.*

(b) *For* $1 \leq k \leq c_n$, $(x_{n,k})_2 = \beta_{n,k}$, $1 \leq k \leq c_n$ *and if* $4i = \varphi(n)$, $(x_{n,i})_2 = \alpha_n$.

(c) $p_n^*(x_{n-1,k}) = 2x_{n,k}$, $1 \leq k \leq c_n$

(d) *Suppose* $k = c_n + 1$ *and* $n \equiv 2$ or 3 (4) *so that* $4k = \varphi(n)$ *and* $H^{4k}(BSO[4k]; Z) \cong Z$, *generated by* λ_k. *Then*

(I) $i_n^*(x_{n,k}) = b_k\lambda_k$, $b_k \in Z$.

(II) $i_n^*(x_{n,k}) = b_k\lambda_k$, $b_k \in Z$.

(III) By (3), $a_k b_k = e_k$, so $a_k, b_k \neq 0$ and both divide e_k. *By* (b) *and* (4), b_k *is odd.*

(Of course, (d) follows from the previous parts.)

Notation. If X and Y are topological spaces, we will write $X \cong Y$ if they are homeomorphic and $X \sim Y$ if they are of the same homotopy type. If G and H are groups, rings, modules, etc., we will write $G \cong H$ if they are isomorphic. If f and g are maps, we write $f \sim g$ if they are homotopic or $f \sim *$ if f is null homotopic. We denote the integers by Z, the integers mod p by Z_p, and the rationals by Q. If $x \in H^n(X; Z)$, then we denote the image of x under the coefficient homomorphism $Z \to Z_p$ by $(x)_p$. A $K(\Pi;n)$ is, as usual, an Eilenberg-MacLane space of type (Π, n). If $\alpha \in H^n(K(\Pi, n); \Pi)$ is the fundamental class (see [10]) and $f:X \to K(\Pi, n)$ is a map, then we call $f^*(\alpha)$ the k-invariant of f.

2. Outline of the construction of the spaces E_α's.

In this section, we will outline the construction of the E_n's and intuitively explain why they have the properties which we want. Unfortunately, although the construction is fairly straightforward, the details are a little involved, and we put this off to section 4. We first discuss the m-connective fiberings over BO, since all spaces are constructed from them.

2.1. We denote by $BO[m]$ the m-connective fibering over BO. Thus, $\pi_i(BO[m]) = 0$, $i < m$ and $\pi_i(BO[m]) \cong \pi_i(BO)$ if $i \geq m$ where this isomorphism is induced by the projection of $BO[m] \to BO$. In particular, $BO[2] = BSO$ so that $BO[m] = BSO[m]$, *for* $m \geq 2$.

Recall if we let $\pi = \pi_m(BO)$, then we may construct $BO[m+1] \to BO[m]$ as the fibration induced from the loop-path fibration over $K(\pi, m)$ by a map $BO[m] \to K(\pi, m)$ (which induces an isomorphism in homotopy in dimension m). By Bott periodicity [2], for $k > 0$,

$$\pi_k(BO) \cong \begin{cases} Z_2, & k \equiv 1, 2 \ (8) \\ Z, & k \equiv 0, 4 \ (8) \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if we let $\varphi(n) = 8a + 2^b$ for $n = 4a + b$, $0 \leq b \leq 3$, then the $(n+1)$-st non-zero homotopy group of BO is in dimension $\varphi(n)$ and so the $(n+1)$-st different $BO[m]$ is $BO[\varphi(n)]$.

In [12], Stong computed $H^*(BO[m]; Z_2)$ and the induced homomorphism $\pi_m{}^*: H^*(BO[\varphi(m)]; Z_2) \to H^*(BO[\varphi(m+1)]; Z_2)$. For applications, one of the most useful of his results is that $\pi_m{}^*$ is 0 (mod 2) in dimensions $< 2^m$ (which is quite a bit more than $\varphi(m+1)$).

2.2 Suppose now that $\varphi(n) \equiv 0$ (4), so that, as noted above, $BO[\varphi(n+1)]$ is induced over $BO[\varphi(n)]$ by a map $f:BO[\varphi(n)] \to K(Z, \varphi(n))$. Let us instead look at the fibration $q_{n+1}: T_{n+1} \to BO[\varphi(n)]$ induced by a non-trivial map $g:BO[\varphi(n)] \to K(Z_2, \varphi(n))$. Then T_{n+1} has the fortunate property that $H^*(T_{n+1}) \cong H^*(BO[\varphi(n+1)]) \otimes H^*(K(Z, \varphi(n))$ (mod 2) and $q^*: H^*(BO[\varphi(n)]) \to H^*(BO[\varphi(n+1)]) \otimes 1$ looks as if it were $\pi_n{}^*$; in particular, it also is 0 in dimension $< 2^n$. This is proven in Lemma 4.2. Furthermore, $g^*: H^{\varphi(n)}(BO[\varphi(n)]; Z) \to H^{\varphi(n)}(T_{n+1}; Z)$ maps $Z \to Z$ and is multiplication by 2, essentially since this is what happens in the fibration $K(Z_2, m-1) \to K(Z, m) \to K(Z, m)$.

Therefore, let $E_1 = BSO$, $E_2 = BO[4]$ and $p_2: E_2 \to E_1$ be the natural fibration with fiber $K(Z_2, n-1)$. Let $E_3 = T_3$ and $p_3 = q_3$, above. Thus

$$H^*(E_3) \cong H^*(BO[8]) \otimes H^*(K(Z, 4)) \qquad \text{(mod 2)}$$

Now, let $p_4 : E_4 \to E_3$ be the fibration induced by a map $E_3 \to K(Z_2, 8) \times K(Z_2, 4)$ which is the g, above, on the $H^*(BO[8]) \otimes 1$ factor of $H^*(E_3)$ and non-zero on the $1 \otimes H^*(K(Z, 4)$ factor. We thus get

$$H^*(E_4) \cong H^*(BO[9]) \otimes H^*(K(Z, 8)) \otimes H^*(K(Z, 4)) \qquad (\text{mod } 2)$$

and p_4^* acts like g_3^* above on the $H^*(BO[8])$ factor and is multiplication by 2 and therefore 0 (mod 2) on the $H^*(K(Z, 4))$ factor. We construct $p_5 : E_5 \to E_4$ by mapping $E_4 \to K(Z_2, 9) \times K(Z_2, 8) \times K(Z_2, 4)$ to be the map which induces $BO[10] \to BO[9]$ on the $H^*(BO[9]) \otimes 1$ factor, and the non-zero map of $K(Z, 4j) \to K(Z_2, 4j)$ on the $H^*(K(Z, 4j))$ factor, and thus get

$$H^*(E_5) \cong H^*(BO[10]) \otimes H^*(K(Z, 8)) \otimes H^*(K(Z, 4)).$$

In general, $p_n : E[n + 1] \to E[n]$ is constructed in exactly one of these two ways. We can keep track of the free generators in $H^*(E_n ; Z)$ using the Pontrjagin classes, the knowledge that $H^*(E_n ; Q)$ is a polynomial algebra and that p_n^* multiplies them by 2, since that is what g^* does, as noted above, as does the projection map in the fibration $K(Z_2, 4t - 1) \to K(Z, 4t) \to K(Z, 4t)$.

3. Computing the geometric dimension

In [3], Gitler and Mahowald construct the following modified Postnikov tower for the fibration

$$V_m \to BSO_m \xrightarrow{q} BSO$$

THEOREM 3.1. *Let $t \leq 2m - 1$. Then there is a sequence of principal fibrations*

$$\cdots \to T_s \xrightarrow{\pi_s} T_{s-1} \to \cdots \to T_1 \xrightarrow{\pi_1} T_0 = BSO$$

together with map $q_s : BSO_m \to T_s$ which are also fibrations with fibers F_s such that
(a) *For $s \geq 1$, $q_{s-1} \sim \pi_s \cdot q_s$ (where $q_0 = q$).*
(b) *For $s \geq 1$, π_s is the pullback from the loop-path fibration over K_{s-1} (by a map $T_{s-1} \to K_{s-1}$, where each K_s is a product of $K(Z_2, i)$'s where $m + 1 \leq i \leq t$*
(c) *Each F_s is min $\{t - 1, m - 1 + q(s)\}$ connected (mod p torsion, $p > 2$) where $q(s)$ is given by the following table.*

	$n \equiv 0$	$n \equiv 1$	$n \equiv 2$	$n \equiv 3$
$s \equiv 0$	$n + 2s - 1, s > 0$	$2s + 1$	$n + 2s - 1, s > 0$	$2s + 1$
$s \equiv 1$	$n + 2s - 2$	$2s$	$n + 2s - 1$	$2s + 1$
$s \equiv 2$	$n + 2s - 2$	$2s - 1$	$n + 2s - 1$	$2s$
$s \equiv 3$	$n + 2s - 3$	$2s + 1$	$n + 2s - 1$	$2s - 1$

where the congruences are taken mod 4.

Pick $t \leq 2m - 1$ and suppose s is big enough so that F_s is $t - 1$ connected. Let n be the largest integer such that $2^n \leq t$. By Theorem 5, $p_{n+1} : E_{n+1} \to E_n$

has the property ${p_{n+1}}^*$ is 0 in dimensions $< 2^{n+1}$ and thus in dimensions $\leq t$. Therefore the composite $p_2 \circ \cdots \circ p_{n+1}: E_{n+1} \to BSO$ also induces the 0 homomorphism in dimensions $\leq t$ so by using 3:1 (b), there is a lifting $f_1: E_{n+1} \to T_1$ of this composite. Similarly, ${p_{n+2}}^*$ is 0 in dimensions $< 2^{n+2}$ so that $(f_1 \circ p_{n+2})^*$ is 0 in dimensions $\leq t$ so there is a lifting $f_2: E_{n+2} \to T_2$ of $f_1 \circ p_{n+2}$. By induction we construct a lifting $f_s: E_{n+s} \to T_s$ of $f_{s-1} \cdot p_{n+s}$. Since F_s is $t - 1$ connected, the image under f_s of the t-skeleton of E_{n+s} lifts to BSO_m. Thus, by construction, if we let $r = n + s$, where $n = |\log_2 t|$ and $s = \min\{s \mid F_s \text{ is } t\text{-connected}\}$, then this r is the smallest r for which the t-skeleton of E_r lifts.

Now let X be a CW complex of dim $\leq t$, x a stable orientable bundle over X, and $g_x: X \to BSO$ a map representing x. We now have the following diagram:

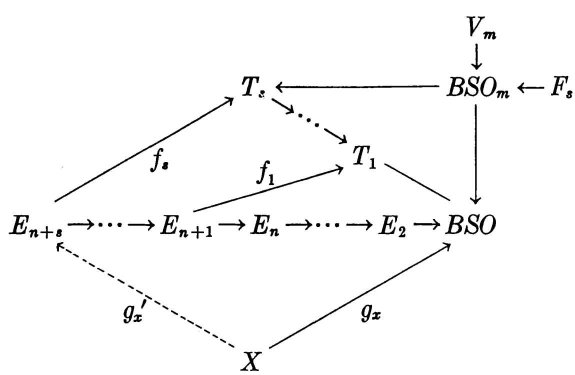


Diagram 3.2

Then, if we can lift g_x to $g_x': X \to E_{n+s}$, we can assume $g_x'[X]$ is contained in the t-skeleton of E_{n+s} so the above lifting of the t-skeleton of E_{n+s} to BSO_m carries $g_x'[X]$ along, i.e., we then get a lifting of g_x to a map $X \to BSO_m$ and consequently codim $(x) \geq t - m$. We now examine some cases that allow us to obtain liftings g_x' and determine the relationships that are Theorems 1 and 2.

We have that for all r, $E_r \to E_{r-1}$ is a principal fibration with fiber a product of $K(Z_2, q_i)$'s with $q_i < \varphi(r-1)$, so that $q_x: X \to BSO$ will lift to $g_x': X \to E_r$ if $H^i(X; Z_2) = 0$ for $0 < i < \varphi(r-1) + 1$. Since $\varphi(m) = 8a + 2^b$ if $m = 4a + b$, $0 \leq b \leq 3$, it is easy to check that

$$q = \varphi(r-1) + 1 = \begin{cases} 2r + 1, & r \equiv 0 \quad (4) \\ 2r \quad, & r \equiv 1 \quad (4) \\ 2r - 1, & r \equiv 2, 3\,(4) \end{cases} \tag{3.4}$$

and we have the following theorem.

THEOREM 3.5. *If* $\dim X = t$, $n = |\log_2 t|$, $s = \min\{s \mid F_s \textit{ is } t\textit{-connected}\}$, *and* $r = n + s$, *then if* q *is the integer given in* 3.4 *and* $H^i(X; Z_2) = 0$, $0 < i < q$, *then every real* k*-plane bundle over* X *with* $k > t$ *has codimension* $\geq t - m$.

Since this is clearly cumbersome, we approximate this and prove Theorem 1.

By above, $q \leq 2r + 1 = 2(n + s) + 1$. By the table in 3.1 (c), $s \leq 1/2(t - m) + 3/2$. Thus $q \leq 2|\log_2 t| + (t - m) + 5$, so that $t - m \geq q - 2|\log_2 t| - 5$. We thus have Theorem 1.

Suppose now that x is a (stable) orientable bundle over X, that $g_1: X \to BSO = E_1$ classifies x, and that there is a lifting $g_{r-1}: X \to E_{r-1}$ of g_1 (so we also have maps $g_s: X \to E_s$ such that $g_s = p_{s+1} \circ g_{s+1}$, $1 \leq s < r - 1$). By Theorem 5, g_{r-1} will lift to $g_r: X \to E_r$ if and only if $g_{r-1}^*(\alpha_{r-1}) = 0 = g_{r-1}^*(\beta_{r-1,i})$, $1 \leq i \leq c_{r-1}$. Now suppose in addition that $H^i(X; Z)$ has no 2-torsion for $i \leq \varphi(r-1) + 1$. By Theorem 5, $(x_{r-1,i})_2 = \beta_{r-1,i}$ for $1 \leq i \leq c_r$ and if $\varphi(r-1) = 4i$, $(x_{r-1,i})_2 = \alpha_{r-1,i}$ so g_{r-1} will lift if and only if $g_{r-1}^*(x_{r-1,i})$ is divisible by 2 for $1 \leq i \leq \varphi(r-1)/4$ and if $g_{r-1}^*(\alpha_{r-1}) = 0$, for some reason, if $\varphi(r-1)$ is not divisible by 4. If $\varphi(r-1) \not\equiv 0(4)$, then $\varphi(r-1) \equiv 1, 2\ (8)$, so unless we have a ξ for which we have some other reason to know that $g_{r-1}^*(\alpha_{r-1})$ is 0, we are forced to assume further that $H^i(X; Z_2) = 0$ for $i \equiv 1, 2\ (8)$ and $i \leq r - 1$. Since $p_{r-1}^*(x_{r-2,i}) = 2x_{r-1,i}$, $1 \leq i \leq \varphi(r-2)/4$, g_{r-1} will lift to g_r, i.e., g_{r-2} will lift to g_r if and only if $g_{r-2}^*(x_{r-2,i})$ is divisible by 4, $1 \leq i \leq \varphi(r-2)/4$ and, if $\varphi(r-1) = 4i$, $g_{r-1}^*(x_{r-1,i})$ is divisible by 2. Continuing like this, we find $g_1: X \to E_1 = BSO$ will lift to $g_1: X \to E_r$ if and only if $g_2^*(x_{2,1})$ is divisible by 2^{r-2}, $g_3^*(x_{3,2})$ is divisible by 2^{r-3}, $g_6^*(x_{6,3})$ is divisible by 2^{r-6}, etc., and in general if $s \leq r - 1$, $s \equiv 2$ or $3\ (4)$ and $i = \varphi(s)/4$, then $g_s^*(x_{s,i})$ is divisible by 2^{r-s}.

Suppose $s \equiv 2$ or $3\ (4)$, so that $\varphi(s)$ is divisible by 4, and that g_1 has a lifting g_s. Let $i = \varphi(s)/4$ and consider $p_i(x)$, the i^{th} Pontrjagin class of x. Let n_i be the power of 2 in $d_i(2x - 1)!$ where $d_i = 1$ or 2 as i is even or odd. Then by [7], the image P_i in $H^*(E_s; Z) = (\text{odd}) \cdot 2^{n_i} x_{s,i}$ so that $P_i(x) = (\text{odd}) 2^{n_i} \cdot f_s^*(x_{s,1})$ and so it is at least divisible by 2^{n_i}. Moreover, for $s \leq r - 1$ so that $i \leq \varphi(r-1)/4$, if $P_i(x)$ is divisible by $2^{n_i + r - s}$, then we have $g_s^*(x_{s,i})$ divisible by 2^{r-s}, which is what we want. Using the formula for $\varphi(n)$ it is easy to check that $s = 2i$ or $2i - 1$ as i is odd or even, and if m_i is the power of 2 in $(2i-1)!$, then $n_i = m_i$ or $m_i + 1$ as i is even or odd. Consequently $n_i + r - s = m_i + 1 + r - 2_i$ in either case. Therefore, if we argue as above, we have the following theorem:

Theorem 3.6. *Suppose* $\dim X = t$, $n = |\log_2 t|$, $s = \min\{s \mid F_s$ *is t-connected*$\}$ $r = n + s$, *and* q *is the integer given in* 3.4. *If* $H^i(X; Z)$ *has no 2-torsion for* $i \leq q$ *and* $H^i(X; Z_2) = 0$ *for* $i \equiv 1, 2\ (8)$ *and* $i < q$, *and if* x *is k-plane bundle over* X *with* $k > t$ *which has its Pontrjagin classes* $P_i(x)$ *divisible by* $2^{m_i + 1 + r - 2i}$ *for* $1 \leq i \leq q/4$, *then codim* $(x) \geq t - m$.

Again, if we approximate in the same way as in the proof of Theorem 1 and using 3.4, noting that $r \leq (q+1)/2$, we have Theorem 2.

4. Details of the construction of the E_n's.

Recall, $H^*(BO; Z_2)$ is a polynomial algebra over Z_2 with generators $w_i \in H^i(BO)$, $i \geq 0$, $w_0 = 1$ (see Milnor [8]). Stong defined, for each $i > 0$, a class $\theta_i \in H^i(BO; Z_2)$ such that $\theta_i \equiv w_i$ modulo products of lower dimensional

w_j's, so that $H^*(BO; Z_2) \cong Z_2[\theta_1, \theta_2, \cdots]$ also. To simplify notation, we shall not distinguish between θ_i in $H^*(BO)$ and its image in the cohomology of any other space and we shall denote $\pi_k(BO)$ by π_k. We now summarize Stong's results in [12].

PROPOSITION 4.1. *Let $p_{n+1}: BO[\varphi(n+1) \to BO[\varphi(n)]$ be the projection. It is induced from the loop-path fibration over a $K(\pi_{\varphi(n)}, n)$ by a map $g_n: BO[\varphi(n)] \to K(\pi_{\varphi(n)}, \varphi(n))$ which induces an isomorphism in homotopy in dimension $\varphi(n)$. For m a positive integer, let $\alpha(m)$ denote the number of ones in the dyadic expansion of m. Then*

(a) $H^*(BO[\varphi(n)]) \cong H^*(K(\pi_{\varphi(n)}, \varphi(n))/I(Q_n e_n) \otimes Z_2[\theta_i \mid \alpha(i-1) \geq n]$, *as rings.*

$$\text{where } Q_n = \begin{cases} Sq^2, & b = 0, 3 \\ Sq^3, & b = 1 \\ Sq^5, & b = 2 \end{cases} \quad \text{for } n = 4a + b, 0 \leq b \leq 3$$

and $I(Q_n e_n)$ denotes the ideal in $H^*(K(\pi_{\varphi(n)}, \varphi(n))$ generalized by $Q_r e_n$ and $0 \neq e_n \in H^{\varphi(n)}(K(\pi_{\varphi(n)}, \varphi(N))$.

(b) *The first factor in* (a) *is contained in $H^*(BO[\varphi(n)])$ as a sub-A algebra, where A is the mod 2 Steenrod algebra.*

(c) *The* Ker $p_{n-1}{}^* = Im\, g_n{}^* =$ *the first factor, above,*

$$\otimes Z_2[\theta_i \mid \alpha(i-1) = n] \text{ and the } I_m p_{n+1}{}^* = Z_2[\theta_i \mid \alpha(i-1) > n].$$

In particular, $p_{n+1}{}^*$ is 0 in dimensions $< 2^{n+1}$ and is onto if and only if $n =$ 0, 1, or 2.

(d) *All the polynomial generators of $H^*(K(\pi_{\varphi(n)}, \varphi(n) - 1))$ transgress in p_{n+1}.*

(e) *Let $\{E_r, d_r\}$ be the* mod 2 *cohomology Serre spectral sequence for p_{n+1}. Then E_∞ is a polynomial algebra with all generators in $E_\infty{}^{0,*}$ or $E_\infty{}^{*,0}$.*

LEMMA 4.2. *Let $n = 4a + b$ where $b = 2$ or 3, so that $\pi = \pi_{\varphi(n)}(BO) \cong Z$. Let $m = \varphi(n) = 8a + 2^b$ and $K = K(Z_2, m)$. Let $\Omega K = K(Z_2, m-1) \to T_{m+1} \xrightarrow{p} BO[m]$ be the fibration induced from the loop-path fibration $\Omega K \to LK \xrightarrow{p_1} K$ by a non-trivial map $f: BO[m] \to K$ (which is unique up to homotopy). Then $H^*(T_{m+1}; Z_2) \cong H^*(BO[\varphi(n+1)]; Z_2) \otimes H^*(K(Z, m); Z_2)$, and this splitting is induced by a map $i: BO[\varphi(n+1)] \to T_{m+1}$ such that $q = pi$.*

Proof. From the long exact homotopy sequence for p and p_i and the map between them induced by f, it follows that $\pi_m(T_{m+1}) \cong Z$, and $p_\#: \pi_i(T_{m+1}) \to \pi_i(BO[m])$ is an isomorphism for $i \neq m$ and is multiplication by 2 in dimension m. Now consider the fibration

$$K(Z, m-1) \to BO[\varphi(n+1)] \xrightarrow{q} BO[m]$$

which, as noted previously, is induced from the loop-path fibration over a

$K(Z, m)$ by a map $f_1{:}BO[m] \to K(Z, m)$ which induces an isomorphism in homotopy in dimension m. Let $r{:}K(Z, m) \to K(Z_2, m)$ be a non-trivial map (which is unique up to homotopy). Consider the following diagram:

$$\begin{array}{ccccccc}
K(Z, m-1) & \longrightarrow & BO[\varphi(n+1)] & \xrightarrow{q} & BO[m] & \xrightarrow{f_1} & K(Z, m) \\
\Big\downarrow{\scriptstyle g|} & & \Big\downarrow{\scriptstyle g} & & \Big\| & & \Big\downarrow{\scriptstyle r} \\
K(Z_2, m-1) & \longrightarrow & T_{m+1} & \xrightarrow{p} & BO[m] & \xrightarrow{f} & K(Z_2, m)
\end{array}$$

Diagram 4.3

Now, by above, $r \circ f_1$ is homotopic to f, and, since $f_1 \circ q$ is nulhomotopic, $f \circ q$ is nulhomotopic. Therefore, there is a lifting of q to $g{:}BO[\varphi(n+1)] \to T_{m+1}$, and, since g covers the identity of $BO[m]$, this g is a map of fibrations. From the long exact homotopy sequences for p and q and from the map between them induced by g, it follows that $g_{\#}{:}\pi_i(BO[\varphi(n+1)]) \to \pi_i(T_{m+1})$ is an isomorphism for $i \neq m$ (since $p_{\#}$ and $q_{\#}$ are isomorphisms for $i \neq m$, and f and f_1 are non-trivial).

Now $\pi_i(T_{m+1}) = 0$ for $i < m$ and $Z \approx \pi_m(T_{m+1}) \approx H_m(T_{m+1}) \approx H^m(T_{m+1}; Z)$. Thus there is a map $h{:}T_{m+1} \to K(Z, m)$ which induces an isomorphism in homotopy in dimension m. Make h into a fibration, call its fiber F, and consider the following diagram:

$$\begin{array}{ccccc}
 & & F & & \\
 & {\scriptstyle \bar{g}} \nearrow & \Big\downarrow{\scriptstyle i} & & \\
BO[\varphi(m+1)] & \xrightarrow{g} & T_{m+1} & \xrightarrow{h} & K(Z, m).
\end{array}$$

Since hg is nulhomotopic, there is a map $\bar{g}{:}BO[\varphi(m+1)] \to F$ such that $\bar{g}i$ is homotopic to g. Since g and i induce isomorphisms in homotopy in all dimensions other than m and $\pi_m(BO[\varphi(m+1)]) = 0 = \pi_m(F)$, $\bar{g}_{\#}$ induces an isomorphism in homotopy in all dimensions. Since all spaces are the same homotopy type as CW complexes (see Milnor [97]) $\bar{g}$ is a homotopy equivalence. i.e., up to homotopy we have

$$BO[\varphi(n+1)] \xrightarrow{g} T_{m+1} \xrightarrow{h} K(Z, m)$$

as a fibration.

We now show $g^*{:}H^*(E_m; Z_2) \to H^*(BO[\varphi(n+1)]; Z_2)$ is epimorphic.

In the case $n = 2$, so $m = \varphi(n) = 4$ and $\varphi(n+1) = 8$, referring to diagram 4.3, q^* is epimorphic, by 4.1, so easily g^* is epimorphic in this case, since $q^* = g^*p^*$.

Suppose now that $n \geq 3$, so that $\varphi(n) \geq 8$. By our restrictions on n,

$$H^*(BO[\varphi(n+1)]) \cong H^*(K(\pi_{\varphi(n+1)}(BO), \varphi(n+1))/I\mathrm{Sq}^2\iota) \otimes Z_2[\theta_i \mid \alpha(i-1) > n+1]$$

where $0 \neq \iota \in H^{\varphi(n+1)}(K(\pi_{\varphi(n+1)}(BO), \varphi(n+1)))$ and $H^*(K(\ \ ,\ \ ))/I$ is contained in $H^*(BO[\varphi(n+1)])$ as a sub-A-algebra. Again, referring to diagram 4.3 and by 4.1, the $1 \otimes Z_2[\theta_i \mid \alpha(i-1) > n+1]$ factor is contained in im q^*, so it is contained in im g^*, similarly to above. Consequently, it is sufficient to find a $\beta \in H^{\varphi(n+1)}(T_{m+1})$ such that $g^*(\beta) = \iota \otimes 1$. Consider the cohomology sequence for h.

(4.4) $\cdots \leftarrow H^{k+1}(T_{m+1}) \xleftarrow{h^*} H^{k+1}(K(Z, m)) \xleftarrow{\delta} H^k(BO[\varphi(n+1)] \xleftarrow{g^*} H^k(T_{m+1})$ $\leftarrow \cdots$ which is exact for $k \leq \varphi(n+1) + \varphi(n) - 2$. Since $\varphi(n+1) > \varphi(n) \geq 8$, this is exact for $k = \varphi(n+1)$, so, to show the existence of a β, it is sufficient to show h^* is $1-1$ in dimension $\varphi(n+1)+1$. Let $0 \neq x \in H^m(K(Z, m); Z_2) \approx Z_2$. By the exactness of 4.4 and since $H^k(BO[\varphi(n+1)]) = 0$ for $k < \varphi(n+1)$, $0 \neq y = h^*(x) \in H^k(T_{m+1}) \simeq Z_2$. By our restrictions on n, $n = 4a + b$, where $b = 2$ or 3. Recall $\varphi(n) = 8a + 2^b$. Consider the cohomology sequence for $p \cdots \xleftarrow{p^*} H^{k+1}(BO[m]) \xleftarrow{\delta} H^k(K(Z_2, m-1)) \xleftarrow{j^*} H^k(T_{m+1}) \xleftarrow{p^*} \cdots$ which is exact for $k \leq 2m - 3$ and therefore for $k = \varphi(n+1) \leq \varphi(n) + 4$. If $0 \neq z \in H^{m-1}(K(Z_2, m-1))$, then $j^*(y) = \mathrm{Sq}^1 z \neq 0$.

Case $b = 3$. Then $\varphi(n+1) = \varphi(n) + 1 = m + 1$, so $\varphi(n+1) = m+2$ and $H^{\varphi(n+1)+1}(K(Z, m)) = \{\mathrm{Sq}^2 x, 0\}$. Therefore, to show h^* is $1-1$, it is sufficient to show $\mathrm{Sq}^2 y \neq 0$. But $j^*(\mathrm{Sq}^2 y) = \mathrm{Sq}^2(j^*(y)) = \mathrm{Sq}^2\mathrm{Sq}^1 z \neq 0$, since $m = \varphi(n) \geq 8$, so $\mathrm{Sq}^2 y \neq 0$.

Case $b = 2$. Then $\varphi(n+1) = \varphi(n)\ 4 = m + 5$, so $\varphi(n+1) + 1 = m + 5$ and $H^{\varphi(n+1)+1}(K(Z, m)) = \{\mathrm{Sq}^5 x, 0\}$. Similarly to the above, $j^*(\mathrm{Sq}^5 y) = \mathrm{Sq}^5\mathrm{Sq}^1 z \neq 0$ (since $m \geq 8$), so $\mathrm{Sq}^5 y \neq 0$.

Since $\pi_{\varphi(n+1)}(BO) \simeq Z$ or Z_2, $H^*(K(\pi_{\varphi(n+1)}(BO), \varphi(n+1); Z_2)$ is a polynomial algebra with $\mathrm{Sq}^I\iota$'s as generators, where the I's are certain admissible sequences (see proof of 4.7), it follows that $H^*(K(\pi_{\varphi(n+1)}(BO), \varphi(n+1))/(I\mathrm{Sq}^2\iota)$ is a polynomial algebra with generators $\{\mathrm{Sq}^{I_j}\iota\}$. Thus $H^*(BO[\varphi(n+1)])$ is a polynomial algebra with $\{\mathrm{Sq}^{I_j}\iota \otimes 1\} \cup \{1 \otimes \theta_i \mid \alpha(i-1) > n+1)$ as a set of generators. Therefore define $\psi: H^*(BO[\varphi(n+1)]) \to H^*(T_{m+1})$ by $\psi(1 \otimes \theta_i) = q^*(1 \otimes \theta_i)$ for $\alpha(i-1) > \mathrm{n} + 1$ (where the second $1 \otimes \theta_i$ here is in $H^*(BO[m])$), (see diagram 4.3) and $\psi(\mathrm{Sq}^{I_j}\iota \otimes 1) = \mathrm{Sq}^{I_j}\beta$, and extend ψ as a ring homomorphism. Since $g^* \circ \psi = 1$, it not only follows that g^* is epimorphic, but we have ψ is a ring-cohomology extension of the fiber in the fibration $BO[\varphi(n+1)] \xrightarrow{g} E_m \xrightarrow{h} K(Z, m)$.

Lemma 4.2 now follows by the Leray-Hirsh Theorem (see Spanier **[10, p. 258]**).

LEMMA 4.5. *Let n, m and $p: T_{m+1} \to BO[m]$ be as in 4.2. Then*
(a) *p^* is zero* (mod 2) *in dimensions* $< 2^{n+1}$.
(b) *$H^m(T_{m+1}; Z) \cong Z$ and $p^*: H^m(BO[m]; Z) \to H^m(T_{m+1}; Z)$ is multiplication by $2: Z \to Z$.*

Proof: (a) Let $f: BO[m] \to K(Z_2, m)$ be the map which induces p, let

$g:BO[m] \to K(Z, m)$ be the map which induces $q:BO[\varphi(n+1)] \to BO[m]$, and let $r:K(Z,m) \to K(Z_2, m)$ be a non-trivial map, so that $rg \sim f$. Then it is sufficient to prove f^* is onto in dimensions $< 2^n$. But, by 4.1, g^* is onto in dimensions $< 2^n$ and $f^* = g^* r^*$ and r^* is onto in all dimensions (mod 2).

(b) From the proof of 4.2, $p_{\#}:\pi_m(T_{m+1}) \to \pi_m(BO[m])$ is multiplication by $2:Z \to Z$. Since T_{m+1} and BO[m] are $(m-1)$ connected, the results follow from the Hurewicz homomorphism, the universal coefficient theorem, and naturality.

LEMMA 4.6. *For $n > 1$ let $K(Z_2, n-1) \to E \xrightarrow{q} K(Z,n)$ be the fibration induced from the loop-path fibration over $K(Z_2, n)$ by a non-zero map $K(Z, n) \to K(Z_2, n)$. Then $E = K(Z, n)$ and both $q_{\#}:\pi_n(K(Z,n)) \to \pi_n(K(Z,n))$ and $q^*:H^n(K(Z, n); Z) \to H^n(K(Z, n); Z)$ are multiplication by $2:Z \to Z$.*

Proof: This is immediate from the long exact homotopy sequence of q, the Hurewicz homomorphism and the universal coefficient theorem.

LEMMA 4.7. *Let $F \xrightarrow{j} E \xrightarrow{\pi} B$ be one of the fibrations*

(1) $K(Z_2, \varphi(n) - 1) \to T_{m+1} \to BO[\varphi(n)]$ of 4.2, *where* $n = 4a + b$, $b = 2, 3$.

(2) $K(_{\varphi(n)}(BO), \varphi(n) - 1) \to BO[\varphi(n+1)] \to BO[\varphi(n)]$ of 4.1.

(3) $K(Z_2, n-1) \to K(Z, n) \to K(Z, n)$ of 4.6.

Then: (a) *The* mod 2 *cohomology of F, E, and B are polynomial algebras.*

(b) *Every polynomial generator of $H^*(F)$ transgresses.*

(c) *If $E = \{E_r^{p,q}, d_r\}$ is the* mod 2 *cohomology spectral sequence of π, then E_∞ is a polynomial algebra with all the generators in $E_\infty^{0,*}$ or $E_\infty^{*,0}$.*

Proof: Case (b): By Adams [1, p. 10],

$$H^*(K(Z_2, n); Z_2) \simeq Z_2[\mathrm{Sq}^I \iota \mid 0 \neq \iota \in H^n(K(Z_2, n)), I = (i_1, \cdots, i_r)$$

is an admissible sequence, excess $I < n$].

By an almost identical proof,

$$H^*(K(Z, n); Z_2) \approx Z_2[\mathrm{Sq}^I \iota \mid 0 \neq \iota \in H^n(K(Z, n)), I = (i_1, \cdots, i_r)$$

is an admissible sequence, excess $I < n$, $i_r \geq 2$ or $I = \emptyset$]. Since $F = K(Z, n)$ or $K(Z_2, n)$ in each of the above cases, and since ι transgresses and Sq's commute with transgression, (b) follows.

Case (a): This case now follows by the proof for (b), by 4.1 and by 4.2.

Case 3 (c). Let $0 \neq \iota_n \in H^n(K(Z, n); Z_2)$ and $0 \neq \iota_{n-1} \in H^{n-1}(K(Z_2, n-1))$.

A quick check shows that $j^*(\iota_n) = \mathrm{Sq}^1 \iota_{n-1} \neq 0$ and that if $\mathrm{Sq}^I \iota_n$ is one of the polynomial generators of $H^*(K(Z, n))$ given above, then $j^*(\mathrm{Sq}^I \iota_n) = \mathrm{Sq}^{I,1} \iota_{n-1}$ and this is one of the polynomial generators of $H^*(K(Z_2, n))$. Therefore, j^* is $1-1$, $E_\infty^{*,*} \cong \operatorname{im} j^*$ which is a sub-polynomial algebra of $H^*(F)$.

Case 2 (c). This is just a restatement of part 4.1.

Case 1 (c): We have a map of fibrations

$$\begin{array}{ccc} K(Z, \varphi(n) - 1) & \xrightarrow{f} & K(Z_2, \varphi(n) - 1) \\ \downarrow & & \downarrow \\ BO[\varphi(n+1)] & \xrightarrow{\bar{f}} & T_{m+1} \\ \downarrow p & & \downarrow \pi \\ BO[\varphi(n)] & \xrightarrow{1} & BO[\varphi(n)] \end{array}$$

This is induced since the k-invariant for π is the mod 2 reduction of the k-invariant for p, and it follows that $f^* \neq 0$. In particular, by the decompositions given in the proof of (b),

$$H^*(K(Z_2, n); Z_2) \cong H^*(K(Z, n); Z_2) \otimes Z_2[\mathrm{Sq}^{I,1}\iota]$$

and f^* is an iso on $H^*(K(Z, n); Z_2) \otimes 1$ factor (and 0 on the $1 \otimes Z_2[\mathrm{Sq}^{I,1}\iota]$ factor). Further, in π, all the $\mathrm{Sq}^{I,1}\iota$'s transgress to 0, by the restriction on n.

By 4.1, the conclusion (c) is true for the fibration p. It follows, by checking elements, that the spectral sequence for π is exactly the spectral sequence for p tensor the polynomial algebra $Z_2[\mathrm{Sq}^{I,1}]$, where of course $1 \otimes Z_2[\mathrm{Sq}^{I,1}] \subset E^{0,*}$. Since the tensor product of polynomial algebras is a polynomial algebra, the result follows.

LEMMA 4.8. *Let R be a field and $F \xrightarrow{i} T \rightarrow B$ be a fibration such that $H^k(F; R)$ is finite and B is the homotopy type of a 1-connected CW complex. Let $E = \{E_r^{*,*}, d_r\}$ be the cohomology spectral sequence of π with coefficients in R, so $E_2^{p,q} \cong H^p(B; R) \otimes H^q(F; R)$. Suppose there are elements $b_j \in H^*(B; R), f_k \in H^*(F; R)$ such that their images freely generate $E_\infty^{*,*}$ as a polynomial algebra,* i.e.,

$$E_\infty^{*,*} \cong R[b_j \otimes 1, 1 \otimes f_k \mid b_j \otimes 1 \in E^{*,0}, 1 \otimes f_k \in E_\infty^{0,*}].$$

Then $H^*(T; R) = R[\pi^*(b_j), t_k \mid j^*(t_k) = f_k]$.

Proof: Let A be the (abstract) polynomial algebra $R[b_j, f_k]$. Grade A, $A = \{A_n\}_{n \geq 0}$ by $b_j \in A_n$ if $b_j \in H^n(B; R), f_k \in A_m$ if $f_k \in H^m(F; R)$ and filter A, $A = F_0A \supset F_1A \supset \cdots$, by $f_k \in F_0A$ (and not in F_1A) for all f_k and $b_j \in F_mA$ (and not in F_{m+1}) if $b_i \in A_m$. Extend both multiplicatively. Take the obvious homomorphism $A \rightarrow H^*(T; R)$ and from the associated bigraded algebra into the spectral sequence. The latter will induce an isomorphism onto $E_\infty^{*,*}$ and the results follow by the 5-lemma.

LEMMA 4.9. *Let B, B' be spaces of the homotopy type of 1-connected CW complexes and let F, F'* be connected. Let *$F \xrightarrow{j} T \xrightarrow{\pi} B$ and $F' \xrightarrow{j'} T' \xrightarrow{\pi'} B'$ be fibrations with* *the following properties*:

(1) *The* mod 2 *cohomologies of* B, B', F, *and* F' *are polynomial algebras, with* $H^*(F;\ Z_2) = Z_2[f_i]$, $H^*(F';\ Z_2) = Z_2[f_i']$, $H^*(B;\ Z_2) = Z_2[b_i]$, *and* $H^*(B';Z_2) = Z_2[b_i']$.

(2) *All the* F_i*'s and* f_i'*'s transgress, and the* f_i*'s are chosen so that if* $\tau(f_i) \neq 0$, *then* $\tau(f_i) \neq \tau(f_j)$ *for* $j \neq i$.

(3) *There are* mod 2 *cohomology ring isomorphisms*

$$k: G^*(B') \to H^*(B), \qquad g: H^*(F') \to H^*(F)$$

with $k(b_i') = b_i$, $g(f_i') = f_i$, *and which commute with the transgressions* τ, τ' *of* π, π', *respectively.*

(4) *If* $E = \{E_r^{p,q}, d_r\}$ *and* $E' = \{E_r'^{p,q}, d_r'\}$ *are the* mod 2 *cohomology spectral sequences of* π *and* π' *respectively, then* $E_\infty^{*,*}$ *is a polynomial algebra with all generators in* $E_\infty^{0,*}$ *or* $E_\infty^{*,0}$.

Then $H^*(T)$ *and* $H^*(T')$ *are polynomial algebras, also, and there is a map of spectral sequences* $\psi = \{\psi_r: E_r' \to E_r \mid r \geq 2\}$ *which is an isomorphism for all* r, *and this induces an* $h: H^*(T'; Z_2) \to H^*(T; Z_2)$ *which is a ring isomorphism and which makes the diagram*

$$\begin{array}{ccccc} H^*(B') & \xrightarrow{\pi'^*} & H^*(T') & \xrightarrow{j'^*} & H^*(F') \\ \downarrow k & & \downarrow h & & \downarrow g \\ H^*(B) & \xrightarrow{\pi^*} & H^*(T) & \xrightarrow{j^*} & H^*(F) \end{array}$$

commute.

Outline of Proof: By the previous lemma, $H^*(T)$ is a polynomial algebra and we can choose classes $t_i \in H^*(T)$, $b_k \in H^*(B)$ so that

$$E_\infty^{0,*} \cong Z_2[1 \otimes j^*(t_i)], \qquad E_\infty^{*,0} \cong Z_2[b_k \otimes 1] \qquad \text{and} \qquad H^*(T) \cong Z_2(\pi^*(b_k), t_i).$$

To construct ψ, the third hypothesis gives a ψ_2 and a fairly straightforward, though tedious, induction yields ψ_r, $r \geq 2$. With ψ_∞ an isomorphism, hypothesis (4) and the previous lemma quickly give the isomorphism h.

Theorem 5, the existence of the spaces E_n and their properties, will follow from the following proposition. For convenience, we restate the properties here, and include a few more which will be needed in the proof.

PROPOSITION 4.10. *Let* $n = 4a + b$, $0 \leq b \leq 3$, $\varphi(n) = 8a + 2^b$, $c_n = \lfloor (\varphi(n) - 1)/4 \rfloor$, *where* $\lfloor x \rfloor$ *is the greatest integer* $\leq x$. *Then there is a sequence of spaces and maps* $p_{n+1}: E_{n+1} \to E_n$, $n \geq 1$ *such that:*

(1) *The map* p_n *is a principal fibration with fiber* $= K(Z_2, \varphi(n) - 1) \times$

$\Pi_{1\le i\le c_n} K(Z_2, 4i - 1)$ *and hence is a rational and* mod p *equivalence, p an odd prime.*

(2) *There is a natural map* $i_n: BO[\varphi(n)] \to E_n$ *such that* $i_n \circ q_{n+1} = p_{n+1} \circ i_{n+1}$, *where* $q_{n+1}: BO[\varphi(n+1)] \to BO[\varphi(n)]$ *is the natural projection.*

(3) *With coefficients* Z_2, $H^*(E_n) \cong H^*(BO[\varphi(n)]) \otimes \{\otimes_{i=1}^{c_n} H^*(K(Z, 4i))\}$ *as rings and the inclusion of* $1 \otimes \{\otimes_i H^*(K(Z, 4i))\}$ *is an A-algebra isomorphism into. Further,* $i_n^*: H^*(BO[\varphi(n)]) \otimes 1 \to H^*(BO[\varphi(n)])$ *is an isomorphism. We will denote by* α_n *and* $\beta_{n,4i}$ *the elements corresponding to the generators of* $H^{\varphi(n)}(BO[\varphi(n)]) \otimes 1$ *and* $1 \otimes \cdots \otimes H^{4i}(K(Z, 4i)) \otimes \cdots \otimes 1$, *respectively.*

(4) Mod 2, *the induced homomorphism* p_n^* *is* 0 *in dimensions* $< 2^n$ *and in fact is* 0 *on the factor* $1 \otimes \{\otimes_i H^*(K(Z, 4i)\}$ *given in* (3).

(5) $H^*(E_n; Q)$ *is a polynomial algebra with one generator in each dimension divisible by* 4. *We can pick* $x_{n,i} \in H^{4i}(E_n; Z)$, $i \ge 1$, which when included into $H^*(E_n; Q)$ *generate the algebra, and which satisfy*

(a) *For* $i \leqq c_n$, $(X_{n,i})_2 = \beta_{n,4i} \in H^{4i}(E_n; Z_2)$ *and if* $\varphi(n) = 4i$, $(x_{n,i})_2 = \alpha_n$.

(b) *For* $i \le \varphi(n)/4$, $p_{n+1}^*(x_{n,i}) = 2x_{n+1,i}$.

Proof: By induction on n we denote by γ_n the generator of

$$H^{\varphi(n)}(BO[\varphi(n)]; Z_2) \cong Z_2.$$

Let $E_1 = BSO = BO[2]$, $E_2 = BO[4]$ and $p_2: E_2 \to E_1$ be the map of 4.1. Recall, $H^*(BSO; Q)$ is a polynomial algebra generated by the Pontrjagin classes $P_i \in H^{4i}$. Since p_2 has fiber a $K(Z_2, 1)$, so also is $H^*(E_2; Q)$. Thus for $n = 2$, we can pick $x_{n,i} \in H^{4i}(E_n; Z)$, $i \ge 1$, which, when included into $H^*(E_n; Q)$, generate the algebra, and we can pick $x_{2,1}$ to generate $H^4(E_2; Z) \cong Z$ and thus its image in $H^4(E_2; Z_2) \cong Z_2$ is γ_2.

These cases now follow from 4.1.

Suppose now we have $p_n: E_n \to E_{n-1}$ satisfying the proposition.

Let $K' = \Pi_{1\le i\le c_n} K(Z_2, 4i)$, $K = K(Z_2, \varphi(n)) \times K'$, and let $k_n: E_n \to K$ be a map with k-invariants α_n and $\beta_{4i,n}$, $1 \le i \le c_n$. Define $p_{n+1}: E_{n+1} \to E_n$ to be the pull back of the loop-path fibration over K. By construction, we have (1).

Case $b = 0$ or 1, so that $\varphi(n) = 8a + 2^b$ is not divisible by 4 and hence $H^{\varphi(n)}(BO[\varphi(n)]; Z) \cong Z_2$. Thus the projection from the loop-path fibration $q_{n+1}: BO[\varphi(n+1)] \to BO[\varphi(n)]$ is induced by a non-trivial map $BO[\varphi(n)] \to K(Z_2, \varphi(n))$ with k-invariant γ_n. But by hypothesis, $i_n^*(\alpha_n) = \gamma_n$, so the composite $BO[\varphi(n+1)] \to BO[\varphi(n)] \to E_n \to K$ is nulhomotopic and we have a lifting $i_{n+1}: BO[\varphi(n+1)] \to E_{n+1}$. By construction, we have (2).

Consider now the fibration

$$K(Z_2, \varphi(n) - 1) \times \Pi_{1\le i\le c_n} K(Z_2, 4i - 1) \to BO[\varphi(n) + 1)] \times \Pi_{1\le i\le c_n} K(Z, 4i) \xrightarrow{\pi} BO[\varphi(n)] \times \Pi_{1\le i\le c_n} K(Z, 4i)$$

which is the product of the fibration q_{n+1} and the fibrations

$K(Z_2, 4i - 1) \to K(Z, 4i) \xrightarrow{\pi_i} K(Z, 4i)$, $1 \leqq i \leqq c_n$. We will show that, mod 2. the fibration p_{n+1} looks like π.

Let $F: E_n \to \Pi_{1 \le i \le c_n} K(Z, 4i)$ be a map with k-invariants $x_{n,i}$, $1 \le i \le c_n$ By hypothesis, $(x_{n,i})_2 = \beta_{r,4i}$, so $F^*: H^*(\Pi K(Z, 4i); Z_2) \to 1 \otimes \otimes H^*(K(Z, 4i); Z_2)$ is an isomorphism. Further, if we take $\Pi_{1 \le i \le c_n} K(Z, 4i) \xrightarrow{1 \le i \le c_n} K' = \Pi_{1 \le i \le c_n} K(Z_2, 4i)$ to be the product of the non-trivial maps $K(Z, 4i) \to K(Z_2, 4i)$, then the composite $E_n \to \Pi K(Z, 4i) \to K'$ has k-invariants $\beta_{n,4i}$, $1 \leqq i \leqq c_n$. Since the composite $E_{n+1} \xrightarrow{p_{n+1}} E_n \xrightarrow{F} \Pi K(Z, 4i) \to K'$ is nulhomotopic and since the product of the fibrations

$$K(Z_2, 4i - 1) \to K(Z, 4i) \xrightarrow{\pi_i} K(Z, 4i), \quad 1 \le i \le c_n$$

is a fibration over $\Pi K(Z, 4i)$ induced from the loop-path fibration over K', we get a lifting $G: E_{n+1} \to \Pi_{1 \le i \le c_n} K(Z, 4i)$. In fact, we get maps of fibrations

$$\begin{array}{ccccc}
\Omega K(Z_2, \varphi(n)) & \xrightarrow{j_1} & \Omega K & \xrightarrow{j_2} & \Omega K' \\
\downarrow & & \downarrow & & \downarrow \\
BO[\varphi(n+1)] & \xrightarrow{i_{n+1}} & E_{n+1} & \xrightarrow{G} & \Pi K(Z, 4i) \\
{\scriptstyle q_{n+1}}\downarrow & & \downarrow{\scriptstyle p_{n+1}} & & \downarrow{\scriptstyle \Pi\pi_i} \\
BO[\varphi(n)]_{i_n} & \longrightarrow & E_n & \xrightarrow{F} & \Pi K(Z, 4i)
\end{array}$$

Diagram 4.11

where the left hand maps come from above. Since $\Omega K \cong \Omega K(Z_2, \varphi(n)) \times \Omega K' =$ fiber of π, $j_1^*: H^*(\Omega K(Z_2, \varphi(n)) \otimes 1 \to H^*(\Omega K(Z_2; \varphi(n))$ and $j_2^*: H^*(\Omega K') \to 1 \otimes H^*(\Omega K')$ are natural isomorphisms and we can define $g: H^*(\Omega K; Z_2) \to H^*$ (fiber of π; Z_2) by $g = j_1^* \otimes (j_2^*)^{-1}$, and this is an A-algebra isomorphism. By hypothesis, part (3), we can define a ring isomorphism

$$k: H^*(E_n; Z_2) \to H^*(BO[\varphi(n)] \times \Pi_{1 \le i \le c_n} K(Z, 4i); Z_2)$$

by $k = i_n^* \otimes (F^*)^{-1}$ so that k is a ring isomorphism on each of the factors $H^*(BO[\varphi(n)]; Z_2) \otimes 1$ and $1 \otimes \cdots \otimes H^*(K(Z; 4i); Z_2) \otimes \cdots \otimes 1$ and is an A-algebra isomorphism on the latter factors. By construction and since $H^*(\Omega K; Z_2)$ and H^*(fiber of π; Z_2) are polynomial algebras on certain $S_q{}^I$'s on the "fundamental classes" (see proof of 4.7), k and g commute with transgression. Let $E = \{E_r, d_r\}$ be the mod 2 cohomology spectral sequence for π. Then E is the tensor product of mod 2 spectral sequences for q_{n+1} and π_i, $1 \leqq i \leqq c_n$, all of which satisfy the hypothesis of 4.9, by 4.7.

Thus, we apply 4.9 to π, p_n, q and k and get

$$h: H^*(E_{n+1}; Z_2) \rightarrow H^*(BO[\varphi(n+1)]; Z_2) \otimes H^*(\Pi_{1\le i\le c_n} K(Z; 4i); Z_2)$$

which is a ring isomorphism and gives us our splitting. By the commutivity of diagram we can take $h = i_{n+1}{}^* \otimes G^{*^{-1}}$ (once we know an isomorphism commuting with the diagram exists). For $1 \le j \le c_n$, we define $x_{n+1,j}$ as the image under G^* of the "fundamental class' in $H^{4j}(\pi_{1\le i\le c_n} K(Z, 4i); Z)$ that comes from the fundamental class in $H^{4j}(K(Z, 4j); Z)$, and we define $\beta_{4j,n+1}$ to be the image of that class reduced mod 2. Thus $p_{n+1}{}^*(x_{n,j}) = 2x_{n+1,j}$, $(x_{n+1,j})_2 = \beta_{4j,n+1}$ and, for $1 \le j \le c_n$, $1 \otimes \cdots \otimes H^*(K(Z, 4j); Z_2) \otimes \cdots \otimes 1$ lies in $H^*(E_n; Z_2)$ as an A-algebra (since G is a map). Note that $c_{n+1} = c_n$ for this case. So at this point, (3) and (4) follow from the above and from 4.5, 4.6, and 4.9. If $b = 0$ (where $n = 4a + b$, $0 \le b \le 3$) so that $H^{\varphi(n+1)}(BO[\varphi(n+1)]; Z) \cong Z_2$, then we pick $\alpha_{n+1} \in H^{\varphi(n+1)}(E_{n+1}; Z_2)$ so that $h(\alpha_{n+1}) = \gamma_{n+1}$. Also, for $i > c_n$, $4i > \varphi(n+1)$ and we can pick $x_{n,i} \in H^{4i}(E_{r+1}; Z)$ that satisfy (5) (essentially since $H^*(E_n; Q)$ is a polynomial algebra). If $b = 1$, $H^{\varphi(n+1)}(BO[\varphi(n+1)]; Z) \cong Z$. For $4i > \varphi(n+1)$, we pick $x_{n,i} \in H^{4i}(E_{n+1}; Z)$ to satisfy (5) as above. Let $i = \varphi(n+1)/4$. Recall, the image of the universal Pontrjagin class $P_i \in H^{4i}(BSO; Z)$ is $\neq 0$ in $H^{4i}(BO[\varphi(n+1)]; Z)$ so its image is $\neq 0$ in $H^{4i}(E_{n+1}; Z)$, since the map $BO[\varphi(n+1)] \rightarrow BSO$ factors $BO[\varphi(n+1)] \xrightarrow{i_{n+1}} E_{n+1} \rightarrow BSO$, by construction and by hypothesis. Let $x_{n+1,i} \in H^{4i}(E_n; Z)$ be the image of P_i, divided by the largest possible integer, so that $x_{n+1,i}$ generates a Z summand in $H^{4i}(E_n; Z)$ and $d_{n+1}{}^*(x_{n+1,i}) \neq 0$. It now follows that all of the $x_{n,j}$'s generate the polynomial algebra $H^*(E_{n+1}; Q)$, since $i_{n+1}{}^*(x_{n,j}) = 0$ for $j < i$ (by connectivity). Let $\alpha_{n+1} = (x_{n+1,i})_2 \in H^{4i}(E_n; Z_2)$, which is $\neq 0$ by the universal coefficient theorem since $x_{n+1,i}$ generates a Z summand. Since $BO[\varphi(n+1)]$ is $(\varphi(n+1) - 1)$ connected and $4c_n < \varphi(n+1)$, any map $BO[\varphi(n+1)] \rightarrow \Pi_{1\le i\le c_n} K(Z, 4i)$ is nulhomotopic. Consequently $G \circ i_{n+1}$ is nulhomotopic so $i_{n+1}{}^* G^* = 0$. By the above splitting, $H^{\varphi(n+1)}(E_n; Z_2) \cong H^{\varphi(n+1)}(BO[\varphi(n+1)]; Z_2) \otimes 1 \oplus 1 \otimes H^{\varphi(n+1)}(\Pi K(Z, 4i); Z_2)$ (since $BO[\varphi(n+1)]$ is $(\varphi(n+1) - 1)$-connected) where the summand is the image of G^* and $i_{n+1}{}^*$ of the first summand is $\neq 0$. It follows that $i_{n+1}{}^*(\alpha_{n+1}) \neq 0$, so it generates $H^{\varphi(n+1)}(BO[\varphi(n+1)]; Z_2) \cong Z_2$ and α_{n+1} lies in the $H^{\varphi(n+1)}(BO[\varphi(n+1)]; Z_2) \otimes 1$ (really, at least has a non-zero component there, but this part of the splitting involves a choice, and we could simply choose α_{n+1}). We have now completed the cases $b = 0$ or 1.

Case $b = 2$ or 3; so that $\varphi(n) = 8a + 2^b$ is divisible by 4 and hence $H^{\varphi(n)}(BO[\varphi(n)]; Z) \cong Z$. Many of the details of these cases are similar to above, and will not be repeated but just referred to. Let $T_{n+1} \xrightarrow{q_{n+1}'} BO[\varphi(n)]$ be the fibration of 4.2, which is induced by a non-trivial map $BO[\varphi(n)] \rightarrow K(Z_2, \varphi(n))$ with k-invariant γ_n. Similarly to the above case, we get a map $i_{n+1}': T_{n+1} \rightarrow E_{n+1}$ such that $p_{n+1} i_{n+1}' = i_n q_{n+1}'$. But by 4.2, there is a map $j_{n+1}: BO[\varphi(n+1)] \rightarrow T_{n+1}$ such that $q_{n+1} = q_{n+1} \circ j_{n+1}$. If we let $i_{n+1} = j_{n+1} \circ i_{n+1}'$, we have (2).

Consider now the fibration

$$K(Z_2, \varphi(n) - 1) \times \Pi_{1 \leq i \leq c_n} K(Z_2, 4i - 1) \to T_{n+1}$$

$$\times \Pi_{1 \leq i \leq c_n} K(Z_2, 4i) \xrightarrow{\pi} BO[\varphi(n)] \times \Pi_{1 \leq i \leq c_n} K(Z, 4i)$$

which is the product of the fibration $q_{n+1'}$ and the fibrations

$$K(Z_2, 4i - 1) \to K(Z, 4i) \xrightarrow{\pi_i} K(Z, 4i), \quad 1 \leqq i \leqq c_n .$$

Let $F: E_n \to \Pi_{1 \leq i \leq c_n} K(Z, 4i)$ be a map with K-invariants $x_{n,i}$, $1 \leqq i \leqq c_n$. Similarly to the above case, we get a lifting $G: E_{n+1} \to \Pi_{1 \leq i \leq c_n} K(Z, 4i)$ and a map of fibrations

$$\begin{array}{ccccc}
\Omega K(Z_2, \varphi(n)) & \xrightarrow{j_1} & \Omega K & \xrightarrow{j_2} & \Omega K' \\
\downarrow & & \downarrow & & \downarrow \\
T_{n+1} & \xrightarrow{i_{n+1}'} & E_{n+1} & \xrightarrow{G} & \Pi K(Z, 4i) \\
\downarrow{\scriptstyle q_{n+1}'} & & \downarrow{\scriptstyle p_{n+1}} & & \downarrow \\
BO[\varphi(n)] & \xrightarrow[i_n]{} & E_n & \xrightarrow[F]{} & \Pi K(Z, 4i)
\end{array}$$

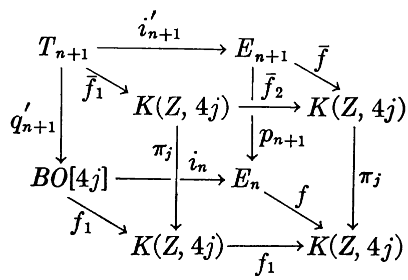
As above, we can define $g: H^*(\Omega K; Z_2) \to H^*(\text{fiber of } \pi; Z_2)$ by $g = j^* \otimes (j_2{}^*)^{-1}$, a natural A-algebra isomorphism and a $k = i_n{}^* \otimes (F^*)^{-1}: H^*(E_n ; Z_2) \to H^*(BO[\varphi(n)] \times \Pi_{1 \leq i \leq c_n} K(Z, 4i): Z_2)$ A-algebra isomorphism on the factors $1 \otimes \cdots \otimes H^*(K(Z, 4i); Z_2) \otimes \cdots \otimes 1$. As above, g and k commute with transgression and if $E = \{E_r, d_r\}$ is the mod 2 cohomology spectral sequence for π, it is the tensor product of the spectral sequence for q_{n+1}' and π_i, $1 \leq i \leq c_n$ all of which satisfy the hypotheses of 4.9, by 4.7. Thus, as above, we apply 4.9 to π, q_{n+1}', g and k and get a ring isomorphism

$$h = i_{n+1}{}^* \otimes (G^*)^{-1}: H^*(E_{n+1} ; Z_2) \to H^*(T_{n+1} ; Z_2) \otimes H^*(\Pi_{1 \leq i \leq c_n} K(Z, 4i); Z_2).$$

For these cases $\varphi(n)/4 = c_{n+1} = c_n + 1$ and since $H^*(T_{n+1}; Z_2) \cong H^*(BO[\varphi(n+1)]; Z_2) \otimes H^*(K(Z; \varphi(n); Z_2)$ as rings (4) follows from the above and from 4.5 and 4.7, and the ring splitting part of (3) follows.

Now similarly to the previous case for $1 \leq i \leq c_n$ we define $x_{n+1,i} \in H^{4i}(E_{n+1} ; Z)$ and $\beta_{4i,n+1} \in H^{4i}(E_{n+1} ; Z_2)$ using G^*, and conclude $(x_{n+1,i})_2 = \beta_{n+1,4i}$, $p_{n+1}{}^*(x_{n,i}) = 2x_{n+1,i}$, and $1 \otimes \cdots \otimes H^*(K(Z, 4i); Z_2) \otimes \cdots \otimes 1$ lies in $H^*(E_{n+1} ; Z_2)$ as a sub A-algebra. We now have one more factor to take care of.

Let $j = \varphi(n)/4$, let $f: E_n \to K(Z, 4j)$ have k-invariant $x_{n,j}$, $f_1: BO[\varphi(n)] \to K(Z, 4j)$ have k-invariant a generator of $H^{\varphi(n)}(BO[4j]; Z) \cong Z$ and let $f_2: K(Z, 4j) \to K(Z, 4j)$ so that $f_2 \circ f_1 = f \circ i_n$ and recall the fibration $K(Z_2, 4_j - 1) \to K(Z, 4j) \xrightarrow{\pi_i} K(Z, 4j)$. We then get the following commutative diagram



Let e be the fundamental class in $H^{4j}(K(Z, 4j); Z)$ and let e_2 be its reduction mod 2. Let $x_{n,j} = \bar{f}^*(e)$, and $\beta_{n,4j} = \bar{f}^*(e_2)$, so that $(x_{n,j})_2 = \beta_{n,4j}$ and $p_{n+1}{}^*(x_{n,j}) = 2x_{n+1,j}$. Let $\epsilon = \bar{f}_1{}^*(e_2)$, so by 4.5, ϵ generates $H^{4i}(T_{n+1}; Z_2) \cong Z_2$. We need to know $i_{n+1}{}'^*(\beta_{n+1,4j}) = \epsilon$. By hypothesis, $(x_{n,j})_2 = \alpha_n$ and $i_n{}^*(\alpha_n) \neq 0$. Thus $f^*(e_2) = \alpha_n$ and $i_n{}^*f^* \neq 0: H^{4j}(K(Z, 4i); Z_2) \to H^{4j}(BO[4j]; Z_2)$. By commutivity, $f_1{}^*f_2{}^* \neq 0$, so $f_2{}^*: H^{4j}(K(Z, 4j); Z_2) \to H^{4j}(K(Z, 4j); Z_2)$. However, $\pi_j\bar{f}_2 = f_2\pi_j$ so that f_2 and $\bar{f}_2$ are the same maps in the sense $f_2{}^* = \bar{f}_2{}^*$. Therefore, $\bar{f}_2{}^* \neq 0$ and since $H^{4j}(K(Z, 4j); Z_2) \cong Z_2$, $\bar{f}_2{}^*(e_2) = e_2$. Consequently, $\epsilon = \bar{f}_1{}^*(e_2) = \bar{f}_1{}^*\bar{f}_2{}^*(e_2) = i_{n+1}{}'^*\bar{f}^*(e_2) = i'^*(\beta_{4j,n+1})$. Now by 4.2, $1 \otimes H^*(K(Z, 4j); Z_2)$ lie in $H^*(T_{n+1}; Z_2)$ as a sub A-algebra and the above ring splitting of $H^*(E_{n+1}; Z_2)$ involves choices of the generators so that they act correctly under $i_{n+1}{}'^*$. Consequently, by the commutivity of the above diagram, we can pick the factor $1 \otimes H^{4j}(K(Z, 4j); Z_2) \otimes \cdots \otimes 1$ in $H^*(E_{n+1}; Z_2)$ to be $f^*H^*(K(Z, 4j); Z_2)$ (which we recall is a polynomial algebra on certain $S_q{}^I$'s on e_2). We thus have this factor also as a sub A-algebra of $H^*(E_{n+1}; Z_2)$.

If $b = 3$ so that $H^{\varphi(n+1)}(BO[\varphi(n + 1)]; Z] \cong Z_2$, then we finish the proof in the same way we did with the case $b = 0$, above.

If $b = 2$ so that $H^{\varphi(n+1)}(BO[\varphi(n + 1)]; Z) \cong Z$, then we proceed in essentially the same way as in the case $b = 1$, above. We have now proven 4.10.

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