ON AXIAL MAPS OF A CERTAIN TYPE

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1. Introduction

Let P^n $(n = 1, 2, \cdots)$ denote real projective n-space, with the usual embeddings $P^1 \subset P^2 \subset \cdots$. A map $u: P^n \rightarrow P^{n+k}$ $(k \geq 1)$ is either homotopic to the constant map or to the inclusion. We describe u as *trivial* in the first case, *non-trivial* in the second. By an *axial map of type* (n, k) we mean a map of $P^n \times P^n$ into P^{n+k} which is non-trivial on both axes. Note that an axial map of type (n, k) determines, by restriction, an axial map of type $(n - 1, k + 1)$, and so forth. Hopf [5] has given necessary conditions for the existence of axial maps, including

Proposition 1.1. *If there exists an axial map of type* (n, k), *where* $n = 2^r$, *then* $k \geq 2^r - 1$.

Sanderson [7] has shown, somewhat indirectly, that an axial map of type (n, k) exists if P^n can be immersed in R^{n+k}. This can be established by direct construction as follows (see §3 for a discussion of the principles concerned). Let X, Y be (smooth) manifolds with tangent bundles $T(X)$, $T(Y)$. An immersion $f: X \rightarrow Y$ determines a monomorphism $T(f): T(X) \rightarrow T(Y)$. We take the direct sum with a trivial line bundle and then pass to the associated projective space bundles. Note that the fibre-preserving map

$$f' = P(T(f) \oplus 1): P(T(X) \oplus 1) \rightarrow P(T(Y) \oplus 1),$$

also respects the canonical cross-sections. In our case $X = P^n$ and a trivialization

$$\theta: P(T(X) \oplus 1) \rightarrow P^n \times P^n$$

can be chosen which transforms the canonical cross-section into the diagonal. Also $Y = R^{n+k}$ and a retraction

$$\rho: P(T(R^{n+k}) \oplus 1) \rightarrow P^{n+k}$$

can be chosen which transforms the canonical cross-section into the constant map. Write

$$\tilde{f} = \rho f' \theta^{-1}: P^n \times P^n \rightarrow P^{n+k}.$$

Then $\tilde{f}$ is non-trivial on one axis and trivial on the diagonal. Hence it follows by elementary homology that $\tilde{f}$ is non-trivial, on the other axis as well, and so $\tilde{f}$ is an axial map. It would be interesting to know under what conditions $\tilde{f}$ has the property that $\tilde{f} \simeq \tilde{f}T$, where $T: P^n \times P^n \rightarrow P^n \times P^n$ denotes the switching map. In particular, does $\tilde{f}$ have this property when f is an embedding? We have not been able to settle this question.

The construction we have given determines a function from the set of regular homotopy classes of immersions of P^n in R^{n+k} to the set of homotopy classes of axial maps of type (n, k). It follows from theorem of Haefliger and Hirsch [4] that this function is surjective when $n < 2k$, and so we obtain

PROPOSITION 1.3. *If there exists an axial map of type (n, k), where $n < 2k$, then there exists an immersion of P^n in R^{n+k}.*

This result is also due to Sanderson [7]. The main purpose of this note is to show that (1.3) is still true when $n \geq 2k$, and thereby establish

THEOREM 1.4. *There exists an immersion of P^n in R^{n+k} $(k \geq 1)$ if and only if there exists an axial map of type (n, k).*

2. Proof of (1.4)

The first step is to establish

LEMMA 2.1. *Suppose that there exists an axial map of type (n, k), where $n \geq 2k$. Then $n \leq 15$.*

For suppose that $n \geq 16$. Then we can choose r, where $r \geq 4$, so that $2^r \leq n < 2^{r+1}$. We distinguish four cases and, under the hypothesis of (2.1), obtain a contradiction in each case. First suppose that $2^r \leq n \leq 2^r + 3$. Then the lemma follows at once from (1.1). Secondly suppose that $n = 2^{r+1} - 1$. By (1.1) of [6] there exists no immersion of P^n in $R^{n+\ell}$, where $\ell = 2^r$. By (1.3), therefore, there exists no axial map of type (n, ℓ), contrary to hypothesis. Thirdly suppose that $2^r + 4 \leq n \leq 2^r + 2^{r-1} + 1$. Choose s, where $1 \leq s \leq r - 2$, so that $2^r + 2^s + 2 \leq n \leq 2^r + 2^{s+1} + 1$. By (3.7) of [2], there exists no immersion of P^m in $R^{m+\ell}$, where $(m, \ell) = (2^r + 2^s + 2, 2^{r-1} + 2^{s+1} - 1)$. By (1.3), therefore, there exists no axial map of type (m, ℓ), contrary to hypothesis. Finally, suppose that $2^r + 2^{r-1} + 2 \leq n \leq 2^{r+1} - 2$. By (3.7) of [2] there exists no immersion of P^m in $R^{m+\ell}$, where $(m, \ell) = (2^r + 2^{r-1} + 2, 2^r + 2^{r-1} - 5)$. By (1.3), therefore, there exists no axial map of type (m, ℓ), contrary to hypothesis. Thus we obtain a contradiction in any case when $n \geq 16$, and (2.1) is established.

Next we recall (see [3]) that P^n can be immersed in R^{n+1} when $n \leq 3$ and when $n = 6, 7$, also that P^n can be immersed in R^{n+k} when (n, k) is one of the ten pairs

$$(4, 3),\ (5, 2),\ (8, 7),\ (9, 6),\ (10, 6),\ (11, 5),\ (12, 6),\ (13, 9),\ (14, 8),\ (15, 7).$$

Thus the proof of (1.4) will be complete when we have established

LEMMA 2.2. *There exists no axial map of type (n, k) if (n, k) is one of the five pairs*

$$(4, 2),\ (8, 6),\ (10, 5),\ (12, 5),\ (13, 8).$$

The first two cases are given at once by (1.1) above. To deal with the third case we recall, from (9.5) of [1], that P^{15} does not admit 9 linearly independent vector fields over P^{10}. By (4.1) of [6], therefore, there exists no axial map of $P^{10} \times P^8$ into P^{15}, hence no axial map of type $(10, 5)$. To deal with the fourth case we recall, from (2.5) of [2], that P^{17} does not admit 13 linearly independent vector fields over P^9. Hence P^{17} does not admit 10 linearly independent vector fields over P^{12}, by (3.3) of [2]. By (4.1) of [6], therefore, there exists no axial map of $P^{12} \times P^9$ into P^{17}, hence no axial map of type (12, 5). To dispose of the fifth case we observe that an axial map of type (13, 8) would determine an immersion of P^{13} in R^{21}, by (1.3), and this would be contrary to (3.8) of [2]. Thus (2.2) is established, and the proof of (1.4) is complete.

[N. B: the following errata concern (2.5) of [2], on which (3.8) of [2] is based. On page 49 of [2], at the end of the enunciation of (2.5), replace $2^r - 4$ by $2^r + 2^s - 4$. On page 50 of [2], at the end of the first sentence of the proof of (2.5), replace $2^r + 2^s - 1$ by $2^r + 2^s$.]

3. Appendix

We restrict our attention to real vector bundles, although the complex case can be treated similarly. If V is a vector bundle then $P(V)$ denotes the associated projective space bundle and $H(V)$ the canonical line bundle over $P(V)$. If L is a line bundle, over the same base as V, then $P(V)$ is naturally equivalent to $P(V \otimes L)$. Hence a trivialization of $V \otimes L$ determines a trivialization of $P(V)$. If $V \otimes L$ is trivial then we describe V as *L-trivial*, in what follows, and note

PROPOSITION 3.1. *The projective space bundle $P(V)$ is trivial if and only if there exists a line bundle L such that V is L-trivial.*

We identify $P(1)$ with X under the projection $\rho: P(1) \to X$. We denote the canonical line bundle of P^n $(n \geq 1)$ by H_n. Let E be an n-plane bundle over X. We have that $\xi \circ P(u) = P(v) \circ \xi$, as shown below, where ξ is given by tensoring with L and u, v are the obvious monomorphisms at the vector bundle level.

$$\begin{array}{ccc} P(1) & \xrightarrow{P(u)} & P(E \oplus 1) \\ \xi \downarrow & & \downarrow \xi \\ P(L) & \xrightarrow{P(v)} & P((E \oplus 1) \otimes L) \end{array}$$

Suppose that $E \oplus 1$ is L-trivial. Choose a trivialization α, say, of $(E \oplus 1) \otimes L$ and consider the composition

$$X \xrightarrow{s} P(E \oplus 1) \xrightarrow{\theta} P^n \times X,$$

where $s = \rho^{-1} \circ P(u)$ and $\theta = P(\alpha) \circ \xi$. By naturality the canonical line bundle $H_n \otimes 1$ over $P^n \times X$ pulls back under $P(\alpha \circ v)$ to the canonical line bundle $H(L)$ over $P(L)$. But $P(\alpha \circ v) = P(\alpha) \circ P(v)$ and $P(\alpha) \circ P(v) \circ \xi = P(\alpha) \circ \xi \circ P(v) = \xi \circ P(u) \circ P(v)$. Since $\xi^* H(L) \approx \rho^* L$ we conclude that

$$s^* \theta^* (H_n \otimes 1) \approx L. \tag{3.2}$$

Now suppose that X is a (smooth) n-manifold with tangent bundle $T(X)$, such that $T(X) \oplus 1$ is L-trivial. Let Y be an $(n + k)$-manifold $(k \geq 1)$ such that $T(Y) \oplus 1$ is trivial. An immersion $f: X \to Y$ determines a monomorphism $T(f): T(X) \to T(Y)$. We write $f' = P(T(f) \oplus 1)$, $g = \varphi f' \theta^{-1}$, as shown in the following diagram, where θ is given by an L-trivialization of $T(X) \oplus 1$ and φ by a trivialization of $T(Y) \oplus 1$.

$$\begin{array}{ccccc} X & \xrightarrow{s} & P(T(X) \oplus 1) & \xrightarrow{\theta} & P^n \times X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{t} & P(T(Y) \oplus 1) & \xrightarrow{\varphi} & P^{n+k} \times Y. \end{array}$$

Here s, t are the canonical cross-section. Applying (3.2) we have at once that $s^*\theta^*(H_n \otimes 1) \approx L$, $t^*\varphi^*(H_{n+k} \otimes 1) \approx 1$. Since the diagram is commutative it follows from these relations that

$$g^*(H_{n+k} \otimes 1) \approx H_n \otimes 1 \oplus 1 \otimes L. \tag{3.3}$$

In particular take $X = P^n$, $Y = S^{n+k}$ (the case of R^{n+k} is similar). Recall that $T(P^n) \oplus 1$ is H-trivial, while $T(S^{n+k}) \oplus 1$ is trivial. Thus an immersion $f: P^n \to S^{n+k}$ determines a map

$$g: P^n \times P^n \to P^{n+k} \times P^n$$

such that

$$g^*(H_{n+k} \otimes 1) \approx H_n \otimes 1 \oplus 1 \otimes H_n.$$

By projecting g onto the left-hand factor we obtain a map of type (n, k), as required.

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