THE CARDINALITY OF ORDINAL EXPONENTIATION

By Alexander Abian

The cardinality of the ordinal sum of a sequence $(r_i)_{i\in w}$ of ordinal type w of ordinals r_i is equal to the cardinal sum of the sequence $(\overline{\overline{r}}_i)_{i\in w}$ of cardinals $\overline{\overline{r}}_i$. The reason for this is the fact that in ordinal arithmetic [1, p. 50] the ordinal sum $\sum_{i\in w} r_i$ of ordinals r_i is defined as the unique ordinal which is similar to the well ordered set obtained by well ordering (in an obvious way) a disjoint union of well ordered sets each similar to r_i. Thus,

$$(1) \qquad \overline{\overline{\sum_{i\in w} r_i}} = \sum_{i\in w} \overline{\overline{r}}_i$$

Therefore, if $r_i > 0$ for every $i\in w$ and if $\sum_{i\in w} r_i$ is infinite then (cf. [2]), we have:

$$(2) \qquad \sum_{i\in w} \overline{\overline{r}}_i = \overline{\overline{w}} \sup_{i\in w} \overline{\overline{r}}_i$$

Since the ordinal product $u\cdot v$ of ordinals u and v is defined [1, p. 51] based on the notion of repeated addition (namely, adding v copies of u), from (1) we have

$$(3) \qquad \overline{\overline{u\cdot v}} = \overline{\overline{u}}\cdot\overline{\overline{v}}$$

Therefore, if $u\cdot v$ is infinite then from (3) it follows

$$(4) \qquad \overline{\overline{u\cdot v}} = \overline{\overline{u}}\cdot\overline{\overline{v}} = \overline{\overline{u}} + \overline{\overline{v}} = \max\{\overline{\overline{u}}, \overline{\overline{v}}\}$$

In sharp contrast to (1) and (3), the cardinality of an infinite ordinal product $\prod_{i\in w} r_i$ of ordinals r_i is not equal, in general, to the infinite cardinal product of the corresponding cardinals $\overline{\overline{r}}_i$. The reason for this is the fact that in ordinal arithmetic the ordinal product $\prod_{i\in w} r_i$ is defined by:

$$(5) \qquad \prod_{i\in w} r_i = \lim_{v\in w} \left(\prod_{i\in v+1} r_i\right) = \bigcup_{v\in w} \left(\prod_{i\in v+1} r_i\right)$$

whereas in cardinal arithmetic the cardinal product $\prod_{i\in w} \overline{\overline{r}}_i$ of cardinals $\overline{\overline{r}}_i$ is defined as the cardinality of the cartesian product of the family $(\overline{\overline{r}}_i)_{i\in w}$.

In this paper we evaluate the cardinality of infinite ordinal product of ordinals. Also, as a Corollary to Theorem 2 and as a contrast to the cardinal arithmetic, we show that *the cardinality of the ordinal product of a nonempty sequence of infinite ordinals is equal to the cardinality of the ordinal sum of that sequence of ordinals.*

In what follows any arithmetical operation among ordinals is performed in the sense of ordinal arithmetic. Thus, if u and v are ordinals u^v is the ordinal exponentiation of u by v and is given by (5) with $w = v$ and $r_i = u$ for every $i \in w$. As usual ω denotes the smallest infinite ordinal.

Lemma 1. *For every infinite ordinal v,*

$$(6) \qquad \overline{\overline{\omega^v}} = \overline{\overline{v}}$$

Proof. Assume on the contrary and let s be the smallest ordinal for which (6) fails.

Since ω^ω is a denumerable ordinal we see that $s > \omega$.

Case 1. Let $s = u + 1$ for some infinite ordinal u. But then

$$\omega^s = \omega^{u+1} = \omega^u \cdot \omega$$

and therefore, by virtue of the choice of s and (4),

$$\overline{\overline{\omega^s}} = \overline{\overline{\omega^u \cdot \omega}} = \overline{\overline{u}} \cdot \omega = \overline{\overline{u}} = \overline{\overline{s}}$$

which contradicts our assumption.

Case 2. Let s be a limit ordinal. But then again by virtue of the choice of s and (4),

$$\overline{\overline{s}} \leq \overline{\overline{\omega^s}} = \overline{\overline{\lim_{\omega<v<s} \omega^v}} \leq {}_s\bigcup_{\omega<v<s} \overline{\overline{v}} \leq \overline{\overline{s}} \cdot \overline{\overline{s}} = \overline{\overline{s}}$$

which again contradicts our assumption.

From the above two cases we see that our assumption is false and the Lemma is proved.

LEMMA 2. *For every infinite ordinal u*

$$\overline{\overline{u^u}} = \overline{\overline{u}} \tag{7}$$

Proof. Let

$$u = v + m \tag{8}$$

where v is a limit ordinal and m is a finite ordinal. But then by (4) we have

$$\overline{\overline{u^u}} = \overline{\overline{u^{v+m}}} = \overline{\overline{u^v}} \cdot \overline{\overline{u^m}} = \overline{\overline{u^v}} \tag{9}$$

Let

$$u = \omega^e n + \cdots + m = v + m \tag{10}$$

where the right side of the first equality represents the normal expansion of u.
Since v is a limit ordinal by [1, p. 61] we have

$$u^v = \omega^{ev} \text{ and therefore } \overline{\overline{u^v}} = \overline{\overline{\omega^{ev}}}$$

which by Lemma 1 implies

$$\overline{\overline{u^v}} = \overline{\overline{e \cdot v}} \tag{11}$$

But then from (10) we obtain

$$\overline{\overline{e}} < \overline{\overline{\omega^e n}} < \overline{\overline{v}} \quad \text{and} \quad \overline{\overline{u}} = \overline{\overline{v}}$$

which in view of (11), (4) and (9) implies (7), as desired.

Theorem 1. *If the product uv of ordinals u and v is infinite and if* $u > 1$ *then*

$$\overline{\overline{u^v}} = \max\{\overline{\overline{u}}, \overline{\overline{v}}\} \tag{12}$$

Proof. Without loss of generality we may assume that both u and v are infinite ordinals.

If $u \leq v$ then $u \leq u^v \leq v^v$ which by (7) implies (12).

If $v \leq u$ then $v \leq u^v \leq u^u$ which by (7) implies (12).

Thus, the theorem is proved.

Below we give rather significant consequences of Theorem 1.

We recall that for every ordinal k, the ordinal ω_k is a cardinal number. Thus, $u \in \omega_k$ if and only if $u < \omega_k$ and if and only if $\overline{\overline{u}} < \omega_k$ for every ordinal u.

Theorem 2. *For every positive ordinal k,*

$$\omega^{\omega_k} = \omega_k \tag{13}$$

Proof. Clearly, $\omega_k \leq \omega^{\omega_k}$ and therefore $\omega_k \subseteq \omega^{\omega_k}$. Thus, to prove (13) it is enough to show that if $u \in \omega^{\omega_k}$ then $u \in \omega_k$. In other words, it is enough to show that if $u < \omega^{\omega_k}$ then $u < \omega_k$.

Let $u < \omega^{\omega_k}$ and let

$$u = \omega^e n + \cdots + m \tag{14}$$

denote the normal expansion of u. Since $u < \omega^{\omega_k}$ we see that $e < \omega_k$ and since ω_k is a cardinal we have

$$e < \omega_k \tag{15}$$

Moreover, since there are finitely many summands in (14) and since the coefficients of powers of ω in (14) are finite, we have $\overline{\overline{u}} = \overline{\overline{\omega^e}}$, where without loss of generality we may assume $e \geq \omega$. But then from (12) and (15) it follows

$$\overline{\overline{u}} = \overline{\overline{\omega^e}} = \overline{\overline{e}} < \omega_k$$

Thus, $u \in \omega_k$, as desired.

Remark. Since $2^\omega = n^\omega = \omega$ for every finite ordinal $n > 1$, in view of (13), we see that ${}_k^{\omega}$ is an ϵ-number (cf. [1, p. 72]). Consequently, for every ordinal k and every ordinal u, we have

$$u^{\omega_k} = \omega_k \quad \textit{provided} \quad 1 < u < \omega_k$$

Since every ϵ-number is also a δ-number and a γ-number (cf. [1, p. 72]), for every ordinal k and every ordinal u we have

$$u + \omega_k = u \cdot \omega_k = u^{\omega_k} = \omega_k \quad \textit{provided} \quad 1 < u < \omega_k \tag{16}$$

We observe however, that in contrast with (16), we have

$$\omega_k + 1 > \omega_k \quad \textit{and} \quad \omega_k \cdot 2 > \omega_k \quad \textit{and} \quad \omega_k^2 > \omega_k$$

Equalities (16) are quite helpful in ordinal arithmetic. For instance (in view of the above observation) they immediately imply that for every ordinal h and k

$$h < k \quad \textit{if and only if} \quad \omega_h + \omega_k = \omega_h \cdot \omega_k = \omega_h^{\omega_k} = \omega_k \, .$$

Finally, we give a generalization of Theorem 1.

THEOREM 2. *If the product* $\prod_{i\in w} r_i$ *of a sequence* $(r_i)_{i\in w}$ *of ordinal type* w *of ordinals* r_i *is infinite and if* $r_i > 1$ *for every* $i \in w$ *then*

$$\overline{\overline{\prod_{i\in w} r_i}} = \max\{\overline{\overline{w}}, \sup_{i\in w} \overline{\overline{r_i}}\} = \overline{\overline{\sum_{i\in w} r_i}} \tag{17}$$

Proof. In view of the hypothesis of the theorem we have

$$\max\{w, \sup_{i\in w} r_i\} \le \prod_{i\in w} r_i \le (\sup_{i\in w} r_i)^w$$

which, in view of (12), implies

$$\overline{\overline{\prod_{i\in w} r_i}} = \max\{\overline{\overline{w}}, \overline{\overline{\sup_{i\in w} r_i}}\}$$

However, $\overline{\overline{\sup_{i\in w} r_i}} \le \overline{\overline{w}} \cdot \sup_{i\in w} \overline{\overline{r_i}}$ which in view of the above and (1) and (2) implies (17), as desired.

COROLLARY. *Let* $(r_i)_{i\in w}$ *be a nonempty sequence of ordinal type* w *of infinite ordinals* r_i. *Then*

$$\overline{\overline{\prod_{i\in w} r_i}} = \overline{\overline{\sum_{i\in w} r_i}} = \overline{\overline{w}} \sup_{i\in w} \overline{\overline{r_i}} \, .$$

IOWA STATE UNIVERSITY, AMES, IOWA.

REFERENCES

[1] H. BACHMANN, Transfinite Zahlen, Springer Verlag, Berlin, 1967.
[2] A. ABIAN, *On the exponentiation of transfinite cardinals* (to appear).