ON THE SINGULAR POINTS OF A GRAPH

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0. Introduction

In this paper we introduce the concept of *singular point of a graph* and prove that any non trivial, bridgeless connected graph contains at least two singular points. This result enables us to characterize cycle-trees as the only graphs without acyclic points that have no singular points.

1. Notation

We shall use Harary's notation [1].

Let G be a graph. $V(G)$ and $E(G)$ are the sets of points and edges of G respectively. Cycles, paths and subsets of $V(G)$ will be considered as subgraphs of G. A point is *adjacent* to a subgraph α of G if it is adjacent to some point of α. It is *α-cyclic* if there is a cycle in α containing it, otherwise it is *α-acyclic*. An edge of G not belonging to α is a *chord* of α whenever both its endpoints belong to α. We denote by $G - \alpha$ the induced subgraph of G whose point-set is $V(G) - V(\alpha)$. A *bridgeless component* of G is a maximal bridgeless connected-subgraph of G. A set $A \subseteq V(G)$ is a *point-cycle* if there exists a cycle α such that $V(\alpha) = A$. A point $u \in V(G)$ is *singular* if there exists a point-cycle A not containing u and having a non empty intersection with every point-cycle containing u.

For any two subgraphs α and β of G, $\alpha +_G \beta$ is the subgraph of G satisfying: $V(\alpha +_G \beta) = V(\alpha) \cup V(\beta)$ and $E(\alpha +_G \beta) = E(\alpha) \cup E(\beta) \cup M$ where M is the set of all edges of G having one endpoint in α and the other in β. Two different points x and y of a cycle γ determine two paths joining x and y: the *arcs xy* of γ.

If α, S and β are subgraphs of G, we will say that S *separates* α from β, with symbols $\langle \alpha, S, \beta \rangle_G$, if each path of G joining a point in α to another point in β has at least one point in S.

G is a *cycle-tree* if it is connected and everyone of its points is contained in exactly one point-cycle.

$| C |$ denotes the cardinality of C.

2. Preliminaries

In this paper it will be useful to refer to the next two lemmas together with corollary 1. The proofs of lemmas 1 and 2 are easy and we shall not give them here.

LEMMA 1. *Let G be a two-connected graph, α a two-connected subgraph of G and K any connected component of $G - \alpha$. Then $\alpha +_G K$ is also two-connected.*

LEMMA 2. *Let G be a two-connected graph, α a connected subgraph of G and K*

a connected component of $G - \alpha$. *Then* $\alpha +_G K$ *is two-connected provided each end-block of* α *contains some point adjacent to* K *which is not a cut-point of* α.

COROLLARY. *Let* G *be a two-connected graph,* L *a non trivial path in* G, *and* K *any connected component of* $G - L$. *If both endpoints of* L *are adjacent to* K, *then* $L +_G K$ *is two-connected.*

3. Principal results

THEOREM 1. *Let* G *be a two-connected graph,* C *a point-cycle of* G *different from* $V(G)$ *and* K *a component of* $G - C$. *Then there exists a (singular) point* $u \in K$ *and a point-cycle* C_0 *of* G *such that:*
i) $u \notin C_0$ *and* C_0 *intersects all point-cycles of* G *containing* u.
ii) $\langle u, C_0, C\rangle_G$.

Proof. The proof is by induction on $N(G) = |E(G)| + |V(G)|$. The simplest case is obtained when $N(G) = 9$. In this case, $G = K_4 - x$ and therefore theorem 1 is true.

Suppose that theorem 1 is true for all two-connected graphs G' with $N(G') < n$ and let G be a two-connected graph with $N(G) = n$, C a point-cycle of G different from $V(G)$ and γ a cycle of G such that $V(\gamma) = C$. The proof is divided into two cases:

Case 1. $G-\gamma$ is not connected.

Let $K_1, K_2, \cdots, K_r$ be the connected components of $G - \gamma$ and suppose $K = K_1$. The graph $G_1 = \gamma +_G K$ is two-connected by lemma 1. Furthermore $N(G_1) < n$. By the induction hypothesis, there exists a point $u \in K$ and a point-cycle C_0 of G_1 such that:
i$'$) $u \notin C_0$ and C_0 intersects all point-cycles of G_1 containing u.
ii$'$) $\langle u, C_0, C\rangle_{G_1}$.

First of all we shall see that u and C_0 satisfy ii). Let $t = (u_0, u_1, \cdots, u_s)$ be a path in G such that $u_0 = u$, $u_s \in C$. Put $m = \min\{i \mid u_i \in C\}$. The path $t' = (u_0, u_1, \cdots, u_m)$ has only one point in C. Recalling that $\langle K_i, C, K_j\rangle_G$ whenever $i \neq j$, one concludes that t' is contained in G_1 and, by ii$'$), that it contains some point of C_0.

Now let A be a point-cycle of G containing u. If A is a point-cycle in G_1, A intersects C_0 by i$'$). If this is not the case, let α be a cycle in G such that $V(\alpha) = A$. Then α contains some chord of γ or some point in K_j with $j \neq 1$. Because of $\langle K, C, K_j\rangle_G$, $j \neq 1$; in both cases, α contains some point $w \in C$. Both uw arcs of α intersect C_0 by ii). This proves i.)

Case 2. $G - \gamma$ is connected.

If K contains some K-acyclic point p, take $C_0 = C$, $u = p$. Suppose now that all points in K are K-cyclic. Let L be an arc of γ with endpoints x and y, of minimal length such that all points of γ adjacent to K are contained in L. Let x' and y' be points of K adjacent to x and y respectively and τ a path in K joining x' and y' which can be assumed to be non hamiltonian (If $\tau = (u_0, u_1, \cdots, u_q)$,

$u_0 = x'$, $u_q = y'$ were hamiltonian, there would be an edge (u_i, u_j) with $i + 1 < j$ because K contains some cycle, and the path $\tau' = (u_0, \cdots, u_i, u_j, \cdots, u_q)$ would be non hamiltonian). Define $G_2 = L +_G K$. By corollary 1, G_2 is two-connected. Furthermore $N(G_2) < n$. Let γ_1 be the cycle formed by L, τ and the edges (x, x') and (y, y') and put $C_1 = V(\gamma_1)$. By the induction hypothesis, there exists a point $u \in K$ and a point-cycle C_0 of G_2 such that

i″) $u \notin C_0$ and C_0 intersects all point-cycles of G_2 containing u.

ii″) $\langle u, C_0, C_1\rangle_{G_2}$

We shall prove that u and C_0 also satisfy ii). Let $t = (u_0, \cdots, u_s)$ be any path in G such that $u_0 = u$, $u_s \in C$. Define t' as in case 1. It is easy to see that t' is contained in G_2. Since $\langle K, L, C\rangle_G$, t' contains some point of L and by ii″) it contains also a point of C_0.

Now let A be a point-cycle of G containing u. If A is a point-cycle of G_2; then by ii″) A intersects C_0. Otherwise A contains some point $w \in C$. Let α be a cycle of G such that $V(\alpha) = A$. By ii) both of the uw-arcs of α intersect C_0.

THEOREM 2. *Let G be a two-connected graph that is not a cycle. Then G contains at least two singular points.*

Proof. Choose a point-cycle A of G with maximum $|A|$ and let α be a cycle of G such that $V(\alpha) = A$.

Case 1. $A = V(G)$.

Since G is not a cycle, α has a chord λ whose endpoints x_1 and x_2 divide α into two arcs α_1 and α_2. Clearly $V(\alpha_1)$ and $V(\alpha_2)$ are themselves point-cycles of G. If we take $C = V(\alpha_1)$ in theorem 1, we obtain one singular point u of G that is an interior point of α_2. Likewise one obtains another singular point of G that is an interior point of α_1.

Case 2. $A \neq V(G)$ and $G - A$ is not connected.

This case is trivial from case 1 in the proof of theorem 1.

Case 3. $A \neq V(G)$ and $G - A$ is connected.

Put $K = G - A$ and let L be an arc of α as chosen for γ in case 2 of theorem 1. Let L' be the other arc of α with endpoints x and y; x' and y' points of K adjacent to x and y respectively and τ any path from x' to y' in K.

L' has a length greater than one because otherwise the cycle γ_1 formed with L, τ and the edges (x, x') and (y, y') would have a length greater than $|A|$. Thus, L' has interior points. Taking $C = V(\gamma_1)$ as in the proof of theorem 1, one obtains a singular point of G that is an interior point of L'. Furthermore, by theorem 1, K contains another singular point of G.

4. Generalizations and applications

THEOREM 1′. Theorem 1 remains valid if we replace "two-connected graph" by "bridgeless connected graph".

Proof. Suppose that G is not two-connected and let B_0 be the block of G containing C.

Case 1. $K \subseteq B_0$.

K contains no cutpoint of G because if $c \in B_0$ were such a point, K would contain all blocks of G different from B_0 and containing c.

Taking $G = B_0$ in theorem 1, one obtains a point $u \in K$ and a point cycle C_0 of B_0 satisfying properties i) and ii.) Since u is not a cutpoint of G, all point-cycles of G containing u are contained in B_0 and theorem $1'$ follows.

Case 2. $K \nsubseteq B_0$.

K contains points of at least one endblock B_t of G different from B_0. Let B' be the block of G preceding B_t in the unique path joining B_0 and B_t in the block-cutpoint tree of G. Moreover let c be the cutpoint contained in both B' and B_t. If B_t is a cycle, take C_0 to be any point-cycle contained in B' and containing c, and u any point in B_t different from c. If B_t is not a cycle, take a point-cycle $C' \neq V(B_t)$ of B_t containing c. Taking $G = B_t$, $C = C'$ in theorem 1 one, obtains a point $u \in V(B_t)$, $u \neq c$, and a point-cycle C_0 of B_t satisfying properties i) and ii). Since all point-cycles of G containing u are included in B_t, they intersect C_0. Furthermore we have obviously $\langle C, C_0, u\rangle_G$. This finishes the proof.

LEMMA 3. *Each endblock of any bridgeless connected graph contains at least one singular point.*

Proof. Let B_t be an endblock of G, c the cutpoint contained in B_t and B' another block containing c. If B_t is a cycle, all points of B_t different from c are singular. If B_t is not a cycle, let u be a singular point of B_t different from c. Obviously u is a singular point of G.

From theorem 1 and lemma 3 we obtain directly the next result.

THEOREM $2'$. *If G is a non trivial bridgeless connected graph that is not a cycle, then it contains at least two singular points.*

Graphs of figure 1 are examples of bridgeless connected graph with exactly two singular points.

In any graph without acyclic points, a point u is singular if and only if it is

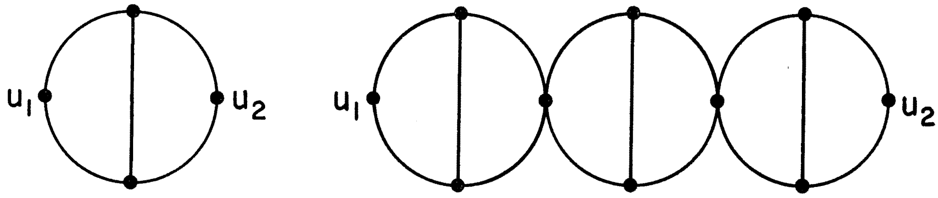

FIGURE 1

singular in the bridgeless component to which it belongs. We obtain then the next theorem.

THEOREM 3. *A graph G without acyclic points has no singular points if and only if it is a cycle-tree.*

It is also clear that if G has no acyclic points and has a singular point, then it has at least one more.

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REFERENCES

[1] F. HARARY, Graph Theory, Addison Wesley, 1971.