MAPPINGS OF QUATERNIONIC PROJECTIVE SPACES

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1. Introduction

Recall that for QP^n, the quaternionic projective space of real dimension $4n$, we have $H^*(QP^n; Z) \cong Z[y]/(y^{n+1})$ where $y \in H^4(QP^n; Z)$ is a generator. To every mapping $f: QP^n \to QP^n$ we can associate an integer $\lambda(f)$, the degree of f by

$$f^* y = \lambda(f) y.$$

We say that an integer λ is *n-realizable* if there exists a mapping $f: QP^n \to QP^n$ such that λ is the degree of f. It is clear from the cellular approximation theorem that n-realizable implies m-realizable for all m, $1 \leq m \leq n$. It is very natural then to consider the following:

Problem. Describe the set of n-realizable integers.

We obtain the following results:

Theorem 1.1. *If λ is n-realizable then*

$$\prod_{i=0}^{k-1} (\lambda - i^2) \equiv 0 \bmod \begin{cases} (2k)! \text{ if } k \text{ is even} \\ (2k)!/2 \text{ if } k \text{ is odd} \end{cases}$$

for all k, $1 \leq k \leq n$.

The results of D. Sullivan [3], together with (1.1) imply

Theorem 1.2. *The set of ∞-realizable integers consists precisely of the odd squares, and zero.*

In [3], Sullivan attributes the necessary condition for ∞-realizability to I. Berstein, R. Stong, L. Smith and G. Cooke.

To our knowledge, none of the proofs appeared in print. We decided to publish this note, because we strongly believe that (1.1) is a characterization of the set of n-realizable integers.

2. Natural endomorphisms

The rings $KU(QP^n)$ are quotient rings of $KU(QP^\infty)$ which is the ring of power series $Z[[z]]$, with $z = H - 1$, where H is the Hopf bundle. On $KU(QP^\infty)$ we have the Adams operations ψ^k which are ring endomorphisms and these restrict to $KU(QP^n) = Z[z]/(z^{n+1})$ in the usual way.

If R is the field of real numbers, the operations ψ^k have a natural extension to ring homomorphisms of $R[[z]]$.

Definition 2.1. *A ring homomorphism φ of $R[[z]]$ will be called a natural endomorphism if $\psi^k \varphi = \varphi \psi^k$ for all k.*

Let us denote by $\mathfrak{N}$ the set of natural endomorphisms of $R[[z]]$. It is clear how to define natural endomorphisms of $R[z]/z^{n+1}$—these we denote by $\mathfrak{N}_n$ and we have the "restriction" maps $\mathfrak{N} \to \mathfrak{N}_n$.

An element φ of $\mathfrak{N}$ is completely determined by its action on the generator z, *i.e.* by the power series $\varphi(z) = \Sigma_i a_i(\varphi) z^i$. If $\varphi \in \mathfrak{N}_n$, then $\varphi(z)$ is just a polynomial, which we call the *characteristic polynomial* of φ.

In general, $\varphi(z)$ is called the *characteristic series* of φ. Let us consider the following map between sets

$$\mathfrak{N} \xrightarrow{\deg} R$$

where $\deg \varphi = a_1(\varphi)$. We now prove the following lemma.

LEMMA 2.2. *The mapping*

$$\deg : \mathfrak{N} \to R$$

is an isomorphism of sets.

Proof. Because φ is a natural homomorphism, φ must commute with ψ^2 and ψ^3. This implies that $a_0(\varphi) = 0$. Note also that the composition of two natural endomorphisms φ, φ' is a natural endomorphism denoted by $\varphi \circ \varphi'$ whose power series is the power series obtained by substitution (as power series) of φ' in φ.

Suppose $\deg \varphi = a_1 \in R$. Then $\varphi(z) = a_1 z + a_2 z^2 + \cdots$. Recall that $\psi^2 z = 4z + z^2$. Since $\varphi\psi^2 = \psi^2\varphi$, we have:

$$\sum_{i>0} a_i(4z + z^2)^i = 4 \sum / _{i>0} a_i z^i + (\sum a_i z^i)^2 \tag{2.3}$$

This equality between the two power series determines all the a_j, $j > 1$ inductively in terms of a_1. This shows that the map deg is injective.

On the other hand, given a number a_1, the equation (2.3) determines uniquely a power series $\varphi(z) = a_1 z + \cdots$, and hence a homomorphism φ which commutes with ψ^2. Now since φ commutes with ψ^2, both $\varphi\psi^k$ and $\psi^k\varphi$ commute with ψ^2, but $a_1(\varphi\psi^k) = a_1(\psi^k\varphi)$, and since a_1 determines a_j uniquely for all $j > 1$, it follows that $\varphi\psi^k = \psi^k\varphi$.

COROLLARY 2.4. *If $\varphi \in \mathfrak{N}$ is not the trivial endomorphism, then φ is in fact an automorphism.*

Proof. Let Θ denote the trivial endomorphism, then

$$\deg|_{\mathfrak{N}-\Theta} : \mathfrak{N} - \Theta \to R - 0$$

is a 1–1 correspondence, but $\deg(\varphi\varphi') = \deg\varphi \deg\varphi'$, *i.e.* it is in fact an isomorphism of groups, so that each φ is invertible, *i.e.*, it is an automorphism.

Note that the arguments would not have changed had we used $\mathfrak{N}_n$ instead of $\mathfrak{N}$. This means that the restriction maps $\mathfrak{N} \to \mathfrak{N}_n$ are 1–1 and onto. For this reason, it suffices to study $\mathfrak{N}$ alone. It is clear that a map $f: QP^n \to QP^n$ induces a natural endomorphism of $KU(QP^n) \otimes R = R[z]/(z^{n+1})$ that is, $f^* \in \mathfrak{N}_n$. It is therefore necessary for n-realizability of an integer k, that $\deg^{-1}(k) \in \mathfrak{N}_n$, have characteristic polynomial with integral coefficients.

It is rather thankless to determine the coefficients $a_j(\varphi)$ from the recursion relation (2.3) and we propose to construct all elements of $\mathfrak{N}$ directly.

Recall that to find the operations ψ^k on $KU(QP^\infty)$ one sets $x + \bar{x} = z = -x\bar{x}$ (in particular $(x+1)(\bar{x}+1) = 1$) and then $\psi^k(z) = (1+x)^k + (1+\bar{x})^k - 2$.

Let μ be any real or purely imaginary number. Consider the power series obtained by setting

$$\varphi_\mu(z) = (1+x)^\mu + (1+\bar{x})^\mu - 2.$$

Since the coefficients of $\varphi_\mu(z)$ are polynomials in μ and $\varphi_{-\mu}(z) = \varphi_\mu(z)$, these polynomials are actually polynomials in μ^2, and hence are real numbers. The power series $\varphi_\mu(z)$ has a nonzero radius of convergence and since

$$[(1+x)^\mu - 1]((1+\bar{x})^\mu - 1) = -[(1+x)^\mu + (1+\bar{x})^\mu - 2]$$

we have:

LEMMA 2.5. *For μ, ν real or purely imaginary numbers,*

$$\varphi_\nu(\varphi_\mu(z)) = \varphi_\mu(\varphi_\nu(z)).$$

Proof. Consider

$$\begin{aligned}\varphi_\nu(\varphi_\mu(z)) &= \varphi_\nu((1+x)^\mu + (1+\bar{x})^\mu - 2)\\ &= [(1+x)^\mu - 1 + 1]^\nu + [(1+\bar{x})^\mu - 1 + 1]^\nu - 2\\ &= (1+x)^{\mu\nu} + (1+\bar{x})^{\mu\nu} - 2\\ &= \varphi_{\mu\nu}(z) = \varphi_\mu(\varphi_\nu(z)).\end{aligned}$$

When k is an integer, $\varphi_k(z) = \psi^k(z)$. Then (2.5) implies that to $\varphi_\mu(z)$ corresponds a unique element in $\mathfrak{N}$ with $\varphi_\mu(z)$ as characteristic series. It is easy to check that $\deg \varphi_\mu = \mu^2$, and so every element of $\mathfrak{N}$ has characteristic series φ_μ for some μ a real or purely imaginary number.

We will now describe $\varphi_\mu(z)$. From the relations

$$x + \bar{x} = z = -x\bar{x}$$

we have formally,

$$\begin{aligned}x &= \tfrac{1}{2}[z + (4z+z^2)^{1/2}]\\ \bar{x} &= \tfrac{1}{2}[z - (4z+z^2)^{1/2}]\end{aligned}$$

and so

$$\begin{aligned}\varphi_\mu(z) &= \left(1 + \frac{z + (4z+z^2)^{1/2}}{2}\right)^\mu + \left(1 + \frac{z - (4z+z^2)^{1/2}}{2}\right)^\mu - 2\\ &= \textstyle\sum_{s>0} \binom{\mu}{s}\left[\left(\frac{z + (4z+z^2)^{1/2}}{2}\right)^s + \left(\frac{z - (4z+z^2)^{1/2}}{2}\right)^s\right]\end{aligned}$$

$$= \sum_{s>0} 2^{-s} \binom{\mu}{s} \sum_{j\geq 0} z^{s-j}[((4z + z^2)^{1/2}) + (-(4z + z^2)^{1/2})^j]$$

$$= \sum_{s>0} \sum_{j\geq 0} 2^{-s+1} \binom{\mu}{s}\binom{s}{2j} z^{s-j}(4 + z)^j$$

$$= \sum_{s>0} \sum_{j\geq 0} \sum_{r\geq 0} 2^{2r-s+1} \binom{\mu}{s}\binom{s}{2j}\binom{j}{r} z^{s-r}.$$

Let us denote the coefficient of z^m in $\varphi_\mu(z)$ by $\alpha_m(\mu)$.
We have

$$\alpha_m(\mu) = \sum_{s\geq m} \sum_{j\geq 0} 2^{s-2m+1} \binom{\mu}{s}\binom{s}{2j}\binom{j}{s-m}. \tag{2.6}$$

Since $\varphi_\mu(z) = \varphi_{-\mu}(z)$, it follows that $\alpha_m(\mu)$ is an even polynomial in μ, *i.e.* $\alpha_m(\mu) = \alpha_m(-\mu)$.

Let

$$P_m(\mu) = \frac{2}{(2m)!} \prod_{i=0}^{m-1} (\mu^2 - i^2) = \binom{\mu + m - 1}{2m - 1}\frac{\mu}{m} \tag{2.7}$$

LEMMA 2.8. *We have:*

$$\alpha_m(\mu) = P_m(\mu).$$

Proof. The terms $\binom{\mu}{s}$ in the expression of $\alpha_m(\mu)$ indicate that $\mu = 0, 1, \cdots, m - 1$ are roots of $\alpha_m(\mu)$, since the summation over s extends over $s \geq m$ only. Since $\alpha_m(\mu)$ is an even polynomial in μ of degree $2m$, we have the roots $\pm 1, \cdots, \pm (m - 1)$ and the double root 0. These are the same as the roots of $P_m(\mu)$. Moreover

$$\alpha_m(m) = \sum_{s\geq m} \sum_{j\geq 0} 2^{s-2m+1} \binom{m}{s}\binom{s}{2j}\binom{j}{s-m}$$

$$= \sum_{j\geq 0} 2^{-m+1} \binom{m}{2j} = 1.$$

Also, clearly, $P_m(m) = 1$. This establishes (2.8).

COROLLARY 2.9. *The polynomials $\psi^k(z)$ are given by*

$$\psi^k(z) = \sum_{m=1}^{k} \binom{k + m - 1}{2m - 1}\frac{k}{m} z^m.$$

This describes the action of the Adams operations in $KU(QP^n)$. Special cases were considered in [1] and [2].

3. Proof of the theorems

Recall that $KSp(QP^\infty) \to KU(QP^\infty)$ is a monomorphism and the image is the submodule generated by z^{2k-1} and $2z^{2k}$, $k = 1, 2, \cdots$. By naturality of this monomorphism (and its restrictions to QP^n) it follows that if f is a map, $f: QP^\infty \to QP^\infty$, then $f^* z \in KU(QP^\infty)$ must lie in the submodule of sympletic elements. If $\deg f = \lambda$, then $f^* z = \varphi_\mu(z)$, with $\mu^2 = \lambda$. It follows that $P_m(\mu)$ must be an integer if m is odd, and an even integer if m is even. For finite dimensional quaternionic projective spaces, this gives theorem (1.1).

For QP^∞, we obtain the conditions

$$(3.1)\qquad \begin{cases} \prod_{i=0}^{m-1} (\lambda - i^2) \equiv 0 \bmod \begin{cases} (2m)! \text{ if } m \text{ is even} \\ (2m)!/2 \text{ if } m \text{ is odd} \end{cases} \\ \text{for all integers } m \geq 1. \end{cases}$$

We wish to show that (3.1) implies that λ must be the square of an odd integer or 0.

If we fix the integer λ, then

$$2 \prod_{i=0}^{m-1} (\lambda - i^2) < 2\lambda^m < (2m)!$$

for large m.

We conclude that (3.1) implies $\lambda = \mu^2$ for some integer μ. This immediately implies that the corresponding characteristic series is in fact a polynomial (in fact $\psi^\mu(z)$). This means that the coefficients are given by $P_m(\mu)$, with μ an integer. From (2.7), we see that if μ were even, and taking $m = \mu$, we would have $P_m(m) = 1$, which is not an even integer. It is therefore necessary that μ be odd.

On the other hand, because $\varphi_\mu = \psi^\mu$ if μ is an integer, $P_m(\mu)$ is an integer and

$$P_m(\mu) = \binom{\mu + m - 1}{\mu - m} \frac{\mu}{m} = \frac{2m}{\mu - m} \binom{\mu + m}{2m} \frac{\mu}{m}$$
$$= \frac{2\mu}{\mu - m} \binom{\mu + m}{2m}.$$

Therefore, if m is even, $\mu - m$ is odd and $P_m(\mu)$ is even. This shows that no more restrictions are imposed on μ. This establishes (1.2).

CENTRO DE INVESTIGACIÓN DEL IPN

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