WEAK EQUIVALENCE OF FIBRATIONS

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Introduction

In this note we introduce the notion of weakly equivalent fibrations which will be used in the study of the group of self equivalences of certain manifolds [2], in a subsequent paper. The main result of this paper is an extension of a theorem of Dold [1], p. 243. We work in the category of spaces of the homotopy type of a countable CW-complex. All fibrations will be assumed to be Serre fibrations and a fibration ξ will be denoted by $\xi = (E(\xi), P(\xi), B(\xi))$ where $E(\xi)$ is the total space of ξ, $B(\xi)$ the base space of ξ, and $P(\xi)$ the projection of $E(\xi)$ onto $B(\xi)$. The author wishes to thank the referee for his helpful comments which resulted in a more readable and concise presentation of this paper.

Definition 1.1 Let ξ and η be two fibrations. A pair of maps (f, g) is called an *equivalence pair from* ξ *to* η if $f{:}E(\xi) \to E(\eta)$, $g{:}B(\xi) \to B(\eta)$ and the following diagram commutes,

$$\begin{array}{ccc} E(\xi) & \xrightarrow{f} & E(\eta) \\ \downarrow{\scriptstyle p(\xi)} & & \downarrow{\scriptstyle p(\eta)} \\ B(\xi) & \xrightarrow{g} & B(\eta) \end{array}$$

FIG. 1

where f and g are homotopy equivalences.

Definition 1.2 Two fibrations ξ and η are called *weakly equivalent* if there exists equivalence pairs (f, g) and (f', g') from ξ to η and from η to ξ respectively, and homotopies $H'{:}E(\eta) \times I \to E(\eta)$, $H{:}E(\xi) \times I \to E(\xi)$, $G'{:}B(\eta) \times I \to B(\eta)$, $G{:}B(\xi) \times I \to B(\xi)$, such that:

(1)
$$\begin{array}{llll} H'_0 = 1_{E(\eta)} & & H_1' = f \circ f' \\ H_0 = 1_{E(\xi)} & & H_1 = f' \circ f \\ G_0' = 1_{B(\eta)} & & G_1' = g \circ g' \\ G_0 = 1_{B(\xi)} & & G_1 = g' \circ g \end{array}$$

and

(2) the following diagrams commute:

$$\begin{array}{ccc} E(\eta) \times I & \xrightarrow{H'} & E(\eta) \\ \downarrow{\scriptstyle p(\eta) \times 1_I} & & \downarrow{\scriptstyle p(\eta)} \\ B(\eta) \times I & \xrightarrow{G'} & B(\eta) \end{array} \qquad \begin{array}{ccc} E(\xi) \times I & \xrightarrow{H} & E(\xi) \\ \downarrow{\scriptstyle p(\xi) \times 1_I} & & \downarrow{\scriptstyle p(\xi)} \\ B(\xi) \times I & \xrightarrow{G} & B(\xi) \end{array}$$

FIG. 2

Note that in the usual notion of "equivalence" of fibrations (fibre homotopy equivalence), g and g' are taken to be the identity map of the base space $B(\eta) = B(\xi)$ and the maps H and H' just cover the identity map of the base spaces.

LEMMA 1.3 *If (f, g) is an equivalence pair from ξ to η, then for all homotopy inverses g' to g there exists $f'\colon E(\eta) \to E(\xi)$ such that f' is a homotopy inverse to f and figure 3 commutes.*

$$\begin{array}{ccccc} E(\xi) & \xrightarrow{f} & E(\eta) & \xrightarrow{f'} & E(\xi) \\ \downarrow{\scriptstyle p(\xi)} & & \downarrow{\scriptstyle p(\eta)} & & \downarrow{\scriptstyle p(\xi)} \\ B(\xi) & \xrightarrow{g} & B(\eta) & \xrightarrow{g'} & B(\xi) \end{array}$$

FIG. 3

Proof: Let f^* be a homotopy inverse to f. (Note that f^* may not even be fibre preserving). Then $p(\xi) \circ f^* \simeq g' \circ g \circ p(\xi) \circ f^* = g' \circ p(\eta) \circ f \circ f^* \simeq g' \circ p(\eta)$. By lifting this composite homotopy, we get $f^* \simeq f'$ such that figure 3 commutes. Since $f^* \simeq f'$ it follows that f' is a homotopy inverse to f.

We now get the extension of the Dold theorem.

THEOREM 1.4 *Let ξ and η be fibrations. Then ξ and η are weakly equivalent if and only if there exists an equivalence pair (f, g) from ξ to η.*

Proof: The "only if" part of the theorem is immediate.

Let (f, g) be an equivalence pair from ξ to η. By Lemma 1.3 there exists an equivalence pair (f', g') from η to ξ such that figure 3 commutes. Since $g \circ g' \simeq 1_{B(\eta)}$ we get a map $h' \simeq f \circ f'$ by the covering homotopy property of η. By Dold [1] there exists a map $g''\colon E(\eta) \to E(\eta)$ with g'' a homotopy inverse to h' and $h' \circ g'' \simeq 1_{E(\eta)}$, with this homotopy covering $1_{B(\eta)}$.

Now let $f^* = f' \circ g''$. Then $f \circ f^* \simeq h' \circ g'' \simeq 1_{E(\eta)}$, with this homotopy covering that of $1_{B(\eta)}$ and $g \circ g'$. We must show that $f^* \circ f \simeq 1_{E(\xi)}$ with the homotopy cover one between $g' \circ g$ and $1_{B(\xi)}$. Repeating the above procedure we find $f^{**}\colon E(\xi) \to E(\eta)$, such that $f^* \cdot f^{**} \simeq 1_{E(\xi)}$.

Then following through the chain of homotopies we have

$$f^* \circ f' \simeq f^* \circ f \circ (f^* \circ f^{**}) \simeq f^* \circ (f \circ f^*) \circ f^{**} \simeq f^* \circ f^{**} \simeq 1_{E(\xi)}.$$

It is easily checked that the homotopy thus arrived at covers a homotopy of $g' \circ g$ and $1_{B(\xi)}$. Thus the result is complete.

From this theorem we get the following useful properties of weak equivalence, and its relation to fibre homotopy equivalence.

COROLLARY 1.5. *If ξ is a fibration, h a homotopy equivalence from B' to $B(\xi)$ then $\eta = h^*(\xi)$, the induced fibration from ξ by h, is weakly equivalent to ξ.*

Proof: Since η is a "pullback" of ξ under the map h, we have the natural map h' from $E(\eta)$ to $E(\xi)$. The map h' is easily seen to be a homotopy equivalence, hence (h', h) will be an equivalence pair from η to ξ. Therefore by Theorem 1.4, it gives rise to a weak equivalence of fibrations.

COROLLARY 1.6. *Let ξ and η be two fibrations which are weakly equivalent by the equivalence pair (f, g) from ξ to η. Then ξ is fibre homotopically equivalent to $g^*(\eta)$.*

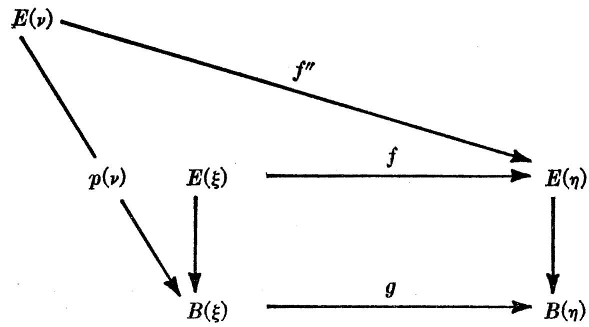

FIG. 4

Proof: Let $g^*(\eta) = \nu$. Consider figure 4 where f'' is the "natural map" from $E(\nu)$ to $E(\eta)$ (f'' is a homotopy equivalence as in Corollary 1.5). By Theorem 1.4 there exists an equivalence pair (f', g') from η to ξ such that (f, g) and (f', g') give rise to a weak equivalence of ξ and η.

We have $p(\xi) \circ (f' \circ f'') = g' \circ g \circ p(\nu)$ and $g' \circ g \simeq 1_{B(\xi)}$ so there exists $h: E(\nu) \to E(\xi)$, with $h \simeq f' \circ f''$ and $p(\xi) \circ h = p(\nu)$, by the covering homotopy property of ξ. Since $h \simeq f' \circ f''$, h is a homotopy equivalence and hence a fibre homotopy equivalence by Dold ([1] Theorem 6.1).

We briefly show that in fact weak equivalence of fibrations is a notion "weaker" than the usual notion of fibre homotopy equivalence. We recall that a space B_F is called a *classifying space*, for fibrations with fibre F if the two functors from a category of topological spaces (for example, CW complexes), $[., B_F]$, and $LF(.)$ are naturally equivalent, where $[., B_F]$ stands for the fibre homotopy equivalence classes of fibrations with fibre F over the given space. Classifying spaces are constructed for example by Stasheff [3].

We have the following proposition for classifying spaces.

PROPOSITION 1.7 *Let B_F be a classifying space for fibrations with fibre F. Let ξ and η be two fibrations over $B = B(\eta) = B(\xi)$ induced by maps f, g into B_F, respectively. Then ξ and η are weakly equivalent if and only if there exists a homotopy equivalence $h: B \to B$ such that $f \circ h \simeq g$.*

Proof: "if" Suppose there exists h such that $f \circ h \simeq g$, h a homotopy equivalence of B; then η is fibre homotopically equivalent to $h^*(\xi)$. But $h^*(\xi)$ is weakly equivalent to ξ by Corollary 1.5 so ξ is weakly equivalent to η.

"only if" If ξ is weakly equivalent to η, then η is fibre homotopically equivalent to $h^*(\xi)$ where (h', h) is an equivalence pair from η to ξ by Corollary 1.6. Using the defining property of classifying spaces we get $f \circ h \simeq g$ and the result follows.

Proposition 1.7 indicates a one to one correspondence between weak equivalence classes of fibrations with fibre F over B and $[B, B_F]/\text{Aut}\ (B)$, where Aut (B) denotes the group of self-equivalences of B.

With this in mind we give a specific example of fibrations weakly equivalent over the same base space which are not fibre homotopically equivalent.

Example: Consider B_{S^k} the classifying space for spherical fibrations with fibre S^k $(k > 1)$. Now consider S^k fibrations over the base space S^1. These are in one to one correspondence with $[S^1, B_{S^k}] \approx \Pi_{1+(k-1)}(S^k) = \Pi_k(S^k) \approx Z$ (see Stasheff [3]).

So taking $h: S^1 \to S^1$ of degree -1 we have that for any non-trivial fibration ξ with fibre S^k over S^1, $h^*(\xi)$ and ξ are weakly equivalent, but not fibre homotopically equivalent, since if f induces ξ then $f \circ h$ induces $h^*(\xi)$ and $f \circ h \not\simeq f$. In fact the weak equivalence classes are gotten from the usual fibre homotopy equivalence classes in this case by identifying fibrations that correspond to $\pm n \in Z \approx [S^1, B_{S^k}]$. This may be more succinctly put by saying that the weak equivalence classes are gotten from the fibre homotopy equivalence classes by dividing out by the action of Eq $(S^1) \approx Z_2$, the group of self equivalences of S^1 (see [2] for some properties of self-equivalences).

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References

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[3] J. Stasheff, *A classification theorem for fiber spaces*, Topology, **2** (1963) 239–46.