A COHOMOLOGY-COMPUTING COVER FOR PSEUDOCONCAVE FAMILIES

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In this paper we will construct an acyclic cover for any family of pseudoconcave analytic spaces, with which we can compute the cohomology of the family for suitable indices. The results are of possible application to the proof of the coherence of direct image sheaves for pseudoconcave families. (For a discussion, see [6].) Before we state the main theorem, we will need some definitions. The notational pattern established here will be used throughout the paper.

Let $\varphi: X \to \boldsymbol{R}$ be a twice continuously differentiable function (hereafter a $\boldsymbol{C}^2$ function) on a domain $X \subset \boldsymbol{C}^n$. Then φ is strongly q-pseudoconvex if, for each $z \in X$, *the complex Hessian*

$$H_\varphi(z) = \left(\frac{\partial^2 \varphi}{\partial z_i \, \partial \bar{z}_j}\right)_z$$

has $n - q + 1$ positive eigenvalues. A real-valued function φ on an analytic space x is strongly q-pseudoconvex if for each z in X there is a neighborhood U of z, a domain D in $\boldsymbol{C}^n$, and an analytic map $h: U \to D$ (called a chart on X) such that h sends U biholomorphically to a closed subvariety of D, and the function $\varphi \circ h^{-1}$ extends to a strongly q-pseudoconvex function Φ on D. For $q = 1$, we say φ is strictly plurisubharmonic.

Let X be an analytic space and, for $z \in X$, *let* ${}_XT_z$ be the space of derivations on the ring $\mathcal{O}_z$ (where $\mathcal{O}$ is the structure sheaf of X). Let $\operatorname{dimt}_z(X)$ be the complex dimension of ${}_XT_z$; we call ${}_XT_z$ the tangent space of X at z, and $\operatorname{dimt}_z(X)$ the tangential dimension. We set $\operatorname{dimt}(X) = \max\{\operatorname{dimt}_z(X) : z \in X\}$.

Remark. For $z \in X$, there is a neighborhood of z biholomorphic to a closed subvariety of a domain in $\boldsymbol{C}^n$ if and only if $n \geq \operatorname{dimt}_z(X)$. (See [3], p. 153.)

Let S be a coherent analytic sheaf on X, and let $h: U \to D$ be a chart for X, with $D \subset \boldsymbol{C}^n$. We can extend the sheaf $h_*(S)$ on $h(U)$ by zero to obtain a coherent sheaf $\hat{S}$ on D. For z in U, there is a neighborhood D' of $h(z)$ and a free resolution

$$0 \to \mathcal{O}^{p_d} \to \cdots \to \mathcal{O}^{p_0} \to \hat{S} \to 0$$

of $\hat{S}$ over D'. Let $d(z)$ be the minimum length needed, and define $dih_z(S) = n - d(z)$. This number is independent of the choices we have made (see [1], p. 196).

Let $f: X \to Y$ again be an analytic map of analytic spaces. Then $f: X \to Y$ is a family of analytic spaces if it has the following local cross-product structure.

For each $z \in X$ there exists a neighborhood U of z, domains D_1 in $\boldsymbol{C}^n$ and D_2

in C^N and charts $h: U \to D_1 \times D_2$ and $\hat{h}: f(U) \to D_2$ such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{h} & D_1 \times D_2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \text{projection}} \\ f(U) & \xrightarrow{\hat{h}} & D_2 \end{array}$$

Further, $f: X \to Y$ is called a q-pseudoconcave family if there exists a function $\varphi: X \to R$ (called an exhaustion function) and real numbers r_*, $r_\#$ in $R \cup \{-\infty, \infty\}$ with $r_* < r_\#$ (called concavity bounds) such that

(1) if $\varphi(z) \in (r_*, r_\#)$, there exists a neighborhood U of z such that $\varphi \mid (U \cap f^{-1}(f(z)))$ is strongly q-pseudoconvex;

(2) for each $r > r_*$ and each compact set $K \subset Y$, the set $f^{-1}(K) \cap \varphi^{-1}([r, \infty))$ is compact.

For $f: X \to Y$ (with $\varphi, r_*, r_\#$) a q-pseudoconcave family, let $X^r = \varphi^{-1}((r, \infty))$, let $X_y = f^{-1}(y)$ and $X \mid B = f^{-1}(B)$, let $X_y^r = X_y \cap X^r$, and let $dih_y^r(S) = \min\{dih_z(S) : \varphi(z) = r \text{ and } f(z) = y\}$.

A q-pseudoconcave family $f: X \to Y$ admits a holomorphic 1-fibering at the points $r \in (r_*, r_\#)$ and $y \in Y$ if there exists a neighborhood $\tilde{X}$ of $\varphi^{-1}(r) \cap X_y = \partial X_y^r$ and analytic families $\tilde{f}: \tilde{X} \to \tilde{Y}$ and $g: \tilde{Y} \to f(\tilde{X})$ such that

1) $$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & f(\tilde{X}) & \end{array}$$ is commutative;

2) if $\operatorname{dimt}_y(Y) = N$ and $\operatorname{dimt}(\tilde{X}) = n + N$, then $\operatorname{dimt}(\tilde{Y}) \leq N + q - 1$;

3) for $\tilde{y} \in \tilde{Y}$, the function $\varphi \mid \tilde{f}^{-1}(\tilde{y})$ is strictly plurisubharmonic.

MAIN THEOREM. *Let $f: X \to Y$ be a q-pseudoconcave family of analytic spaces with exhaustion function φ and concavity bounds r_*, $r_\#$. Suppose the family $f: X \to Y$ admits a holomorphic 1-fibering at the points $r \in (r_*, r_\#)$ and $y \in Y$. Then for S a coherent analytic sheaf on X, there is a Stein open neighborhood B of y, a finite collection $P_i(t)$ of holomorphic polyhedra with $P_i \subset X_y$, and a number $T \in (0, 1)$ such that, for any $t \in [T, 1]$,*

$$\bar{X}^r \mid B \subset \textstyle\bigcup_i \{P_i(t) \times B\} \quad \textit{and} \quad H^j(X^r \mid B, S) \cong H^j(\{P_i(t) \times B\}, S)$$

for $j < dih_y^r(S) - \operatorname{dimt}_y(Y) - q$.

1. Our method of proof will be to extend certain intermediate results in Andreotti and Grauert ([1]), constructing a special open cover with which we can compute the cohomology of $S \mid X^r$. In this section we will prove the key proposition of the paper; after it is completed, we will extend the result to families to obtain a proof of the main theorem.

PROPOSITION 1. *Let X be a domain in C^n $(n \geq 2)$, and let V be a closed analytic subvariety of X. Let $\varphi: V \to \mathbf{R}$ be a continuous function on V, and let $\xi_1, \cdots, \xi_k$ be points in $V \cap \varphi^{-1}(0)$.*

Assume for each ξ_i there is a neighborhood U_i of ξ_i and a C^2 strictly plurisubharmonic function $\varphi_i: U_i \to R$ such that $\varphi_i \mid U_i \cap V = \varphi \mid U_i \cap V$. Let X^0 be the set $\{z \in \bigcap_i U_i: \varphi_i(z) > 0$ for at least one $i\}$. Then for each point ξ_i there is a fundamental system $\{P_i\}$ of Stein neighborhoods of ξ_i in U_i such that:

(I) $H^0(P_1 \cap \cdots \cap P_k, \mathcal{O}) \cong H^0(P_1 \cap \cdots \cap P_k \cap X^0, \mathcal{O})$ *if* $k < n$;

(II) $H^j(P_1 \cap \cdots \cap P_k \cap X^0, \mathcal{O}) = 0$ *for* $0 < j < n - k$.

Proof. Take a particular ξ_i, an extension φ_i in U_i, and assume $\xi_i = 0 \in C^n$. Let φ_i be called φ. The Taylor expansion of φ at 0 can be written

$$\varphi(z) = 2Re(\Sigma_j \varphi_j(0) z_j + \Sigma_{i,j} \varphi_{ij}(\delta) z_i z_j) + \Sigma_{i,j} \varphi_{i\bar{j}}(\delta) z_i \bar{z}_j$$

where

$$\varphi_j(x) = \left(\frac{\partial \varphi}{\partial z_j}\right)_x, \quad \varphi_{ij}(x) = \left(\frac{\partial^2 \varphi}{\partial z_i \, \partial z_j}\right)_x, \quad \varphi_{i\bar{j}}(x) = \left(\frac{\partial^2 \varphi}{\partial z_i \, \partial \bar{z}_j}\right)_x,$$

and $\delta = rz$ for some $r \in [0, 1)$.

As H_φ is positive definite near 0, there is a neighborhood U of 0 and a constant $a > 0$ such that, for $z, z', z'' \in U$, we have

$$H_\delta(z)(z' - z'') \geq a \| z' - z'' \|$$

where $\| z \| = \sup_i | z_i |$. Using the uniform continuity of the second derivatives, we can shrink U so that

$$| \varphi_{ij}(z) - \varphi_{ij}(z') | < \frac{a}{2n^2}$$

for $z, z' \in U$ and $1 \leq i, j \leq n$. Let

$$f(z) = \Sigma_j \varphi_j(0) z_j + \Sigma_{i,j} \varphi_{ij}(0) z_i z_j ;$$

if $f \not\equiv 0$ on U, then the above inequalities guarantee that, on the analytic hypersurface $\{f = 0\} \cap U$, we have $\varphi(z) \geq \frac{1}{2} a \| z \|^2 > 0$ for $z \neq 0$. If $f \equiv 0$, then $\varphi(z) > \varphi(0)$ for all $z \in U - \{0\}$, so we can take any analytic function f which is zero at 0, to conclude that on $\{f = 0\}$, we have $\varphi(z) > \varphi(0)$. Shrink each coordinate function in C^n so that we may assume $\{\| z \| < 1\}$ is relatively compact in U.

Choose $\epsilon > 0$ sufficiently small that the following are satisfied:

(1) For all $c \in C$ with $| c | < \epsilon$, the boundary $\partial\{f(z) = c, \| z \| < 1\}$ of the set $\{f(z) = c, \| z \| < 1\}$ is contained in X^0;

(2) For each $\alpha = 1, \cdots, n$, the set

$$\{z \in U : |f(z)| < \epsilon, \tfrac{1}{2} < |z_\alpha| < 1, |z_i| < 1 \text{ for } i \neq \alpha\}$$

is relatively compact in X^0.

Let $z_0(z) = (2f(z) - \epsilon)/(2\epsilon - f(z))$; then we have $\{z : |f(z)| < \epsilon\} = \{z : |z_0(z)| < 1\}$ and $\{z : f(z) = \frac{1}{2}\epsilon\} = \{z : z_0(z) = 0\}$. By our choice of f and U, we have $\varphi(z) > 0$ on the set $\{z \in U : f(z) = \frac{1}{2}\epsilon\}$; hence there exists a constant $0 < b < 1$ such that the open set $\{z : |z_0(z)| < b, \|z\| < 1\}$ is relatively compact in X^0. Let $P = \{z \in U : |z_0(z)| < 1, |z_i| < 1 \text{ for } i = 1, \cdots, n\}$.

Let $\xi_1, \cdots, \xi_k$ be points as in the statement of the proposition, with extension functions $\varphi_1, \cdots, \varphi_k$. Construct a polyhedron P_i around each point ξ_i with respect to φ_i; let $P = P_1 \cap \cdots \cap P_k$ and assume $P \neq \emptyset$. Further assume $k < n$. The polyhedron P is defined as

$$P = \{z \in U_1 \cap \cdots \cap U_k : |z_i^j| < 1 \text{ for } j = 1, \cdots, k \text{ and } i = 0, \cdots, n\}$$

where $P_j = \{z \in U_j : |z_i^j| < 1 \text{ for } i = 0, \cdots, n\}$. The set $\{z_i^j; j = 1, \cdots, k$ and $i = 0, \cdots, n\}$ is a collection of $k(n+1)$ holomorphic functions on $U = U_1 \cap \cdots \cap U_k$ satisfying the following conditions:

(i) $\{\frac{1}{2} < |z_\alpha^\beta| < 1\} \cap P \Subset X^0$ for each $\alpha \in \{1, \cdots, n\}$ and each $\beta \in \{1, \cdots, k\}$;
(ii) $\{|z_0^j| < b_j \text{ for } j = 1, \cdots, k\} \cap P \Subset X^0$;
(iii) The map $z \to (z_i^j(z))$ is a biholomorphic map of P onto a closed subvariety of the polydisk $\{\|w\| < 1\}$ in $C^{k(n+1)}$.

Take $\delta > 0$ sufficiently small that, for each $j = 1, \cdots, k$, the set

$$P_{j,\delta} = \{|z_i^j| < 1 + \delta \text{ for } i = 0, \cdots, n\}$$

is still contained in U_j.

Let Δ_j be the compact subset of $P_{j,\delta}$ defined as

$$\Delta_j = \{z \in P_{j,\delta} : \varphi_j(z) \leq 0 \text{ and } |z_i^j| \leq 1 \text{ for } i = , \cdots 0, n\};$$

by the results of Narasimhan ([4]), we see that each Δ_j coincides with its proper envelope $\hat{\Delta}_j$ with respect to $P_{j,\delta}$. Hence we can construct a decreasing sequence $Q_{j,s}$ of polyhedra in $P_{j,\delta}$ such that $Q_{j,s+1} \Subset Q_{j,s}$ and $\bigcap_s Q_{j,s} = \Delta_j$. Each $Q_{j,s}$ is of the form

$$Q_{j,s} = \{z \in P_{j,\delta} : |f_{j,s,i}| < \tfrac{1}{2} \text{ for } i = 1, \cdots, r_{j,s}\};$$

without loss of generality we may assume $|f_{j,s,i}| < 1$ on $P_{j,\delta}$ for each i.

Let $P_\delta = P_{1,\delta} \cap \cdots \cap P_{k,\delta}$, let $\Delta = \Delta_1 \cap \cdots \cap \Delta_k$, let $Q_s = Q_{1,s} \cap \cdots \cap Q_{k,s}$, and let $r_s = r_{1,s} + \cdots + r_{k,s}$. Let $W_s = P - (\overline{P \cap Q_s})$; this is an increasing sequence of open sets in P_δ such that $\bigcup_s W_s = P \cap X^0$.

For $D(1)$ the polydisk of radius 1 in $C^{k(n+1)+r_s}$, we have a map $t_s : P \to D(1)$ defined by

$$t_s(z) = (z_1^1(z), \cdots, z_n^k(z), f_{s,1}(z), \cdots, f_{s,r_s}(z), z_0^1(z), \cdots, z_0^k(z)).$$

Take a strictly increasing sequence $\{R_s\}$ of real numbers such that $R_1 > \max\{\frac{1}{2}, b_1, \cdots, b_k\}$ and $\lim_{s\to\infty} R_s = 1$. Choose each R_s sufficiently large that, for $D(R_s)$ the polydisk of radius R_s, we have:

(a) $t_s(P) \cap D(R_s) \neq \emptyset$;
(b) $t_s^{-1}(D(R_s)) \Subset t_{s+1}^{-1}(D(R_{s+1}))$;
(c) $\bigcup_s t_s^{-1}(D(R_s)) = P$.

For a particular s, look at the domains

$$A_\alpha = \{\tfrac{1}{2} < |w_\alpha| < 1\} \cap D(R_s);\ \alpha \in \{1, \cdots, k\cdot n + r_s\}$$

$$A_0 = \{|w_{k\cdot n+r_s+j}| < b_j \text{ for } j = 1, \cdots, k\} \cap D(R_s)$$

and let $Z_s = \bigcup_{\alpha=0}^{kn+r_s} A_\alpha$. We have guaranteed that $t_s(W_s) \cap D(R_s) \subset Z_s$ and that $M_s = t_s^{-1}(Z_s) \subset P \cap X^0$. The set $\{M_s\}$ is an increasing sequence of open sets such that $\bigcup_s M_s = P \cap X^0$. Let s be fixed, and remove it from the notation.

Let I be the sheaf of ideals defined in $D(1)$ by the analytic subvariety $t(P)$. By Proposition 1 of [1], we have a free sheaf resolution

$$0 \to \mathcal{O}^{p_d} \to \cdots \to \mathcal{O}^{p_0} \to I \to 0$$

over $D(R)$. We know that $dih(I) = k(n+1) + r - d$; if $dih(t(P))$ is defined to be $dih(I) - 1$, then, by Proposition 3 of [1], we also have that $dih(t(P)) = dim(t(P)) = n$; hence $d = (k-1)(n+1) + r$.

Lemma 1 of [1] states that, for Z of this form, $H^i(Z, \mathcal{O}) = 0$ for $i \neq 0, k\cdot n + r$. From the appropriate large rectangular diagram, we can then conclude that

$$H^i(Z, I) \cong H^{i+d}(Z, \mathcal{O}^{p_d})$$

for $i > 0$ such that $kn + r > i + d = i + (k-1)(n+1) + r$, in other words, for $0 < i < n - k + 1$. As $k < n$, we get in particular that $H^1(Z, I) = 0$. Hence we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & H^0(D(R), I) & \to & H^0(D(R), \mathcal{O}) & \to & H^0(P \cap t^{-1}(D(R)), \mathcal{O}) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^0(Z, I) & \to & H^0(Z, \mathcal{O}) & \to & H^0(M, \mathcal{O}) & \to 0 \end{array}$$

Both rows are exact, and the first two vertical arrows are isomorphisms by Hartogs Theorem (see [3], p. 21); hence the last vertical arrow is also an isomorphism. Note that this implies $M \neq \emptyset$.

From the exact sheaf sequence

$$0 \to I \to \mathcal{O} \to \mathcal{O}/I \to 0$$

we get the long exact cohomology sequence

$$\cdots \to H^i(Z, \mathcal{O}) \to H^i(M, \mathcal{O}) \to H^{i+1}(Z, I) \to \cdots$$

where the outside groups are both zero for $0 < i < n - k$. Thus $H^i(M, \mathcal{O}) = 0$ for $0 < i < n - k$.

In conclusion, we have constructed an increasing sequence $\{M_s\}$ of open sets in P such that

(a) $\bigcup_s M_s = P \cap X^0$, $\bar{M}_s$ is compact, and $\bar{M}_s \subset M_{s+1}$ for all s sufficiently large;

(b) $H^0(t_s^{-1}(D(R_s)), \mathcal{O}) = H^0(M_s, \mathcal{O})$;

(c) $H^j(M_s, \mathcal{O}) = 0$ for $0 < j < n - k$.

We are now able to apply Proposition 9 of [1] to obtain the desired conclusion.

Remark. The proof of this proposition is essentially that of Lemma 2 (p. 222) of [1]; all we have done is extend it to intersections of neighborhoods constructed with respect to different strictly plurisubharmonic extensions.

2. We now extend the result of section 1 to a q-pseudoconcave family $f: X \to Y$. For $y \in Y$, let $N = \operatorname{dimt}_y(Y)$ and let $n = \operatorname{dimt}(\partial X_y^r)$, where $\partial X_y^r = \{z \in X_y : \varphi(z) = r\}$. We assume that f factors as $g \circ \tilde{f}$ near ∂X_y^r, where $\tilde{f}: \tilde{X} \to \tilde{Y}$ and $g: \tilde{Y} \to f(\tilde{X})$ are analytic families as in the statement of the main theorem. We can cover ∂X_y^r by open sets U_j as follows:

Let $\Delta_1 = \{w \in \mathbf{C}^{n-q+1} : \| w \| < 1\}$

$\Delta_2 = \{w \in \mathbf{C}^{q-1} : \| w \| < 1\}$

$\Delta_3 = \{w \in \mathbf{C}^{N} : \| w \| < 1\}$

For each point $z \in \partial X_y^r$ there is a neighborhood U_z of z in X and chart mappings h_z, $\tilde{h}_z$, $\hat{h}_z$ such that

$$\begin{array}{ccc} U_z & \xrightarrow{\;h_z\;} & \Delta_1 \times \Delta_2 \times \Delta_3 \\ {\scriptstyle \tilde{f}}\big\downarrow & & \big\downarrow{\scriptstyle \text{proj}} \\ \tilde{f}(U_z) & \xrightarrow{\;\tilde{h}_z\;} & \Delta_2 \times \Delta_3 \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \text{proj}} \\ f(U_z) & \xrightarrow{\;\hat{h}_z\;} & \Delta_3 \end{array}$$

is commutative.

Further, the function $\varphi \circ h_z^{-1}$ is the restriction of a $\mathbf{C}^2$ function Φ on $\Delta_1 \times \Delta_2 \times \Delta_3$ such that, on each fiber $\Delta_1 \times \{w_1\} \times \{w_2\}$, Φ is strictly plurisubharmonic.

For S a coherent analytic sheaf on X, let $\hat{S}$ be the trivial extension of the sheaf $h_z^*(S)$ (on $h_z(U_z)$) to all of $\Delta_1 \times \Delta_2 \times \Delta_3$. For U_z sufficiently small, $\hat{S}$ has a resolution

$$0 \to \mathcal{O}p^{P_d} \to \cdots \to \mathcal{O}p^{P_0} \to \hat{S} \to 0$$

over $\Delta_1 \times \Delta_2 \times \Delta_3$, where $d = N + n - dih_z(S)$. As ∂X_y^r is compact, we can take a finite subcover $\{U_i : i = 1, \cdots, s\}$ of the cover $\{U_z\}$.

Lemma. *If $U_i \cap U_j \neq \varnothing$, then for each $z \in U_i \cap U_j$ there are neighborhoods V_i^z and V_j^z of $h_i(z)$ and $h_j(z)$ in $\Delta_1 \times \Delta_2 \times \Delta_3$, and a biholomorphic map $h_{ij}; V_i^z \to V_j^z$ which extends the map $h_j \circ h_i^{-1}: V_i^z \cap h_i(U_i \cap U_j) \to V_j^z \cap h_j(U_i \cap U_j)$. Further, we can guarantee h_{ij} is fiber-preserving.*

Proof. The first part is Satz 17, p. 26 of [5]. We can guarantee that h_{ij} preserves fibers by starting in Δ_3, and extending step-wise.

Take a refinement $\{W_j\}$ of the cover $\{U_i\}$ which has the following property: For each W_j there is a $U_{s(j)}$ such that the set $St(W_j, \{W_i\}) = \bigcup \{W_k : W_k \cap W_j \neq \emptyset\}$ is contained in $U_{s(j)}$. (This is called a star-refinement of $\{U_i\}$; see for example, [2], p. 167).

For each $z \in \partial X_y^r$, choose a W_j such that $z \in W_j$, and look at $h_{s(j)}(z)$. By the above lemma, we can move a neighborhood $V_{s(j)}^z(k)$ of $h_{s(j)}(z)$ biholomorphically to a neighborhood of $h_z(z)$ whenever $z \in U_k$. Let $V_{s(j)}^z$ be the intersection of the neighborhoods $V_{s(j)}^z(k)$ for all k such that $z \in U_k$. By shrinking $V_{s(j)}^z$, we can assume $h_{s(j)}^{-1}(V_{s(j)}^z) \subset W_j$. Call $V_{s(j)}^z$ just V^z.

For each $z \in \partial X_y^r$, construct a polyhedron $P^z \subset V^z$ of the following sort: $P^z = P_0^z \times \Delta_2' \times \Delta_3'$ where P_0^z is a polyhedron centered at $h(z)$ in the 1-pseudoconcave fiber through $h(z)$, constructed as in section 1, and where Δ_2', Δ_3' are sufficiently small polydisks in Δ_2, Δ_3 so that, for each $w_2 \in \Delta_2'$ and each $w_3 \in \Delta_3'$, the functions defining $P_0^z \times \{w_2\} \times \{w_3\}$ in $\Delta_1 \times \{w_2\} \times \{w_3\}$ satisfy the same properties (1) and (2) required of the functions defining P_0^z.

Let $P_1, \cdots, P_s$ be a finite collection of these polyhedra such that $\{h^{-1}(P_i)\}$ covers ∂X_y^r. If $h^{-1}(P_1) \cap \cdots \cap h^{-1}(P_k) \neq \emptyset$ (where h is the appropriate chart map), then we know that $h^{-1}(P_1) \cup \cdots \cup h^{-1}(P_k) \subset U_j$ for some j. Hence $z_1, \cdots, z_k \in U_j$, so we can move each P_i to the copy of $\Delta_1 \times \Delta_2 \times \Delta_3$ associated to the chart $h_j : U_j \to \Delta_1 \times \Delta_2 \times \Delta_3$. We now assume we have $P_1, \cdots, P_k \subset \Delta_1 \times \Delta_2 \times \Delta_3$, with $P = P_1 \cap \cdots \cap P_k \neq \emptyset$. Each P_i is constructed with respect to an extension φ_i of φ. If we let $P^r = P \cap \{\varphi_i > r$, for at least one $i\}$, then we have the following extension of proposition 1:

PROPOSITION 2. (I) $H^0(P, \mathcal{O}) \cong H^0(P^r, \mathcal{O})$ *if* $0 < n - q - k + 1$; (II) $H^j(P^r, \mathcal{O}) = 0$ *for* $0 < j < n - q - k + 1$.

Proof. This is Proposition 12 of [1], except that we apply the proof to intersections as well, using proposition 1.

PROPOSITION 3. *Let S be the coherent analytic sheaf from above; then*
(I) $H^0(h^{-1}(P), S) \cong H^0(h^{-1}(P) \cap X^r, S)$ *if* $0 < \operatorname{dih}_y^r(S) - N - q - k + 1$;
(II) $H^j(h^{-1}(P) \cap X^r, S) = 0$ *for* $0 < j < \operatorname{dih}_y^r(S) - N - q - k + 1$.

Proof. This is Theorems 9 and 10 of [1], again applied to intersections.

Cover X_y^r by open sets $P_i \subset X^r$ such that P_i is biholomorphic to a closed subvariety of $D(1) \times \Delta_i$, where $D(1)$ is the polydisk of radius 1 in some $\boldsymbol{C}^m$ and Δ_i is a Stein neighborhood of y in Y. Then the sets $\{P_i\} \cup \{h^{-1}(P_j)\}$ cover the compact set $\bar{X}^r \cap f^{-1}(\bar{B})$ for B a sufficiently small Stein neighborhood of y. Take a finite subcover of the cover $\{P_i \mid B\} \cup \{h^{-1}(P_j) \mid B\}$ and call it $\{P_1, \cdots, P_s\}$. By including $\boldsymbol{C}^l$ in $\boldsymbol{C}^{l+k}$ in a nice fashion, we can assume we have a map $g_i : P_i \to D(1) \times B$ for $D(1) \subset \boldsymbol{C}^m$, with m independent of i, and with $g_i(P_i)$ a closed analytic subvariety. Further, there exists an $R_0 < 1$ such that, for any $R \in [R_0, 1]$, the sets $P_i(R) = g_i^{-1}(D(R) \times B)$ still cover $\bar{X}^r \mid B$ and still satisfy the conclusions of Proposition 3.

PROPOSITION 4. *Let B^1 be a Stein open set in B, and let $R \in [R_0, 1]$; then*

$$H^j(X^r \mid B^1, S) \cong H^j(\{P_i(R) \mid B^1\}, S)$$

for

$$j < dih_y{}^r(S) - N - q.$$

Proof. We wish to show that $H^j(X^r \mid B^1, S) \cong H^j(\{X^r \cap P_i(R) \mid B^1\}, S) \cong H^j(\{P_i(R) \mid B^1\}, S)$ for $j < dih_y{}^r(S) - N - q$. By (II) of proposition 3, the first isomorphism is a special case of Leray's Theorem for acyclic covers (the construction of the desired isomorphism requires the vanishing of cohomology precisely where we have shown it).

Result (I) of proposition 3 shows that

$$C^k(\{X^r \cap P_i(R) \mid B^1\}, S) \cong C^k(\{P_i(R) \mid B^1\}, S)$$

for $k < dih_y{}^r(S) - N - q$; as the restriction isomorphisms are compatible with coboundary maps, we get the second isomorphism.

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REFERENCES

[1] A. ANDREOTTI AND H. GRAUERT, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math France **90** (1962), 193–259.

[2] J. DUGUNDJI, Topology, Boston, Mass., Allyn and Bacon, 1966.

[3] R. C. GUNNING AND H. ROSSI, Analytic Functions of Several Complex Variables, Englewood Cliffs, N. J., Prentice-Hall, Inc , 1965.

[4] R. NARASIMHAN, *The Levi problem for complex spaces*, Math. Annalen, **142** (1961), 355–65.

[5] H. OELJEKLAUS, *Einbettung komplexer Räume in komplex-projektive Räume*, Schr. Math. Inst. Univ. Münster, **23** (1962).

[6] Y. T. SIU, *A pseudo-concave generalization of Grauert's direct image theorem*, Ann. Scuola Norm. Sup. Pisa, **24** (1970), 279–330 and 439–89.