ON SUMS OF INDEPENDENT RANDOM VARIABLES DEFINED ON A BINARY TREE*

BY A. JOFFE AND A. R. MONCAYO

Introduction and an example

Suppose that a certain variety of seed gives rise to a plant which produces a random number Z of new seeds and then dies; these seeds are blown away randomly by the wind. Each one of these new seeds gives rise to a plant and each plant produces a random number Z of new seeds, then dies, and so on. Assume that the climatic conditions are constant throughout the time and that each seed is blown independently of the others and with the same distribution to a new position on the ground.

For Galton-Watson process Harris [1] conjectured that the distributions of the spread of these plants properly normalized converge in mean square to a normal distribution. This conjecture was proved by P. Ney [2] for a similar model. We intend to show that mean square convergence can be replaced by almost everywhere convergence and moreover we have reason to believe that such convergence holds conditionally on the tree of the Galton-Watson process. This motivates the study of sums of random variables on non random trees, and in this paper we illustrate the idea of proof in the simplest situation of the binary tree; our purpose here is essentially methodological.

Model. Given a "tree" $\{X(\delta_1\delta_2 \cdots \delta_n)\}_{n=1,2,\ldots}$ and $\delta = 0$ or 1, of i.i.d. random variables with mean zero and variance one, defined on a probability space $(\Omega, \mathfrak{F}, P)$, let $S(\delta_1\delta_2 \cdots \delta_n)$ be the sum of the r.v.'s being on the branch $\delta_1\delta_2 \cdots \delta_n$, that is $S(\delta_1\delta_2 \cdots \delta_n) = X(\delta_1) + X(\delta_1\delta_2) + \cdots + X(\delta_1\delta_2 \cdots \delta_n)$. We are interested in the study of the limiting behaviour of the sequence of random distributions which are determined for each n by the 2^n random points $\{n^{-1/2}S(\delta_1\delta_2 \cdots \delta_n)\}_{\delta_1\delta_2\ldots\delta_n}$, each of them weighing 2^{-n}. We have found (theorem) that in this particular tree the limiting distribution is normal with mean zero and variance one. For each n its random distribution function can be expressed as

$$\eta_n(x, \omega) = 2^{-n}\Sigma_n V_x(n^{-1/2}S(\delta_1\delta_2 \cdots \delta_n, \omega)),$$

where for each real number x, V_x is the real function defined by $V_x(y) = 1$ if $y < x$ and $= 0$ if $x \leq y$, and the symbol Σ_n denotes summation over all the indices $\delta_1, \delta_2, \cdots, \delta_n$. We shall be consistent in the use of this notation for the rest of the paper. Its (random) characteristic function is

$$\psi_n(t, \omega) = 2^{-n}\Sigma_n \exp\{itn^{-1/2}S(\delta_1\delta_2 \cdots \delta_n, \omega)\}$$

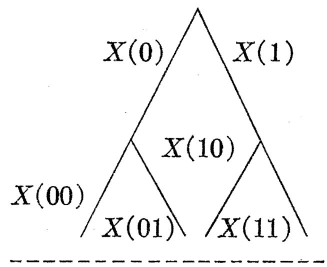
Lemmas 1 and 2 combine to prove (Lemma 3) that for each fixed number t a.s. (ω) the sequence $\{\psi_n(t, \omega)\}$ converges to $e^{-t^2/2}$, and by virtue of Lemma 4 we

* This research was supported by CIMASS (UNAM) and the Canadian National Research Council.

conclude that a.s. (ω) for almost all (Lebesgue) t we have convergence of $\psi_n(t, \omega)$ to $e^{-t^2/2}$, which proves the

THEOREM. $\eta_n(x, \cdot) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$ a.s. (ω) *when* $n \to \infty$.

In the following computations it may help to keep in mind the figure



We shall denote by $\phi(\cdot)$ the characteristic function of $X(0)$ and by $\mathcal{L}$ the lebesgue measure on the real line R.

LEMMA 1. *For each* t, *and* $\epsilon < 2^{-1}$ *an arbitrary positive number*

$$\sup_{1 \le p \le n^{2^{-1}-\epsilon}} |\psi_{n+p}(t, \cdot) - \psi_n(t, \cdot)| \to 0 \text{ a.s. } (\omega) \text{ when } n \to \infty.$$

Proof. First observe that for $p = 1, 2, \cdots, [n^{2^{-1}-\epsilon}]$,

$$\begin{aligned} \psi_n(t, \cdot) &= 2^{-n-p}\Sigma_n 2^p \exp\{itn^{-1/2}S(\delta_1\delta_2 \cdots \delta_n)\} \\ &= 2^{-n-p}\Sigma_{n+p} \exp\{itn^{-1/2}S(\delta_1\delta_2 \cdots \delta_n)\}, \end{aligned}$$

hence

$$\begin{aligned} |\psi_{n+p}(t, \cdot) - \psi_n(t, \cdot)| &\le 2^{-n-p}\Sigma_{n+p}|\exp\{it(n+p)^{-1/2}S(\delta_1\delta_2 \cdots \delta_{n+p})\} \\ &\quad - \exp\{itn^{-1/2}S(\delta_1\delta_2 \cdots \delta_n)\}| \\ &\le 2^{-n-p}\Sigma_{n+p}|\exp\{it(n+p)^{-1/2}S(\delta_1\delta_2 \cdots \delta_{n+p}) - itn^{-1/2}S(\delta_1\delta_2 \cdots \delta_n)\} - 1|, \end{aligned}$$

and in view of the inequality $|1 - e^{i\alpha}| \le |\alpha|$ which holds for any real number α,

$$\begin{aligned} &\le 2^{-n-p}\Sigma_{n+p}t|(n+p)^{-1/2} \\ &\qquad \cdot\{S(\delta_1\delta_2 \cdots \delta_n) + X(\delta_1\delta_2 \cdots \delta_{n+1}) + \cdots + X(\delta_1\delta_2 \cdots \delta_{n+p})\} \\ &\quad - n^{-1/2}S(\delta_1\delta_2 \cdots \delta_n)| \\ &= 2^{-n-p}\Sigma_{n+p}t|\{(n+p)^{-1/2} - n^{-1/2}\}S(\delta_1\delta_2 \cdots \delta_n) + (n+p)^{-1/2} \\ &\qquad \cdot\{X(\delta_1\delta_2 \cdots \delta_{n+1}) + \cdots + X(\delta_1\delta_2 \cdots \delta_{n+p})\}| \le a_n + b_n, \end{aligned}$$

where

$$\begin{aligned} a_n &= t2^{-n}\Sigma_n \frac{(n+p)^{1/2} - n^{1/2}}{n^{1/2}(n+p)^{1/2}} \{|X(\delta_1)| + |X(\delta_1\delta_2) + \cdots + |X(\delta_1\delta_2 \cdots \delta_n)|\}, \\ b_n &= t2^{-n-p}\Sigma_{n+p}(n+p)^{-1/2}\{|X(\delta_1\delta_2 \cdots \delta_{n+1})| + \cdots + |X(\delta_1\delta_2 \cdots \delta_{n+p})|\}. \end{aligned}$$

Now observe that

$$\begin{aligned} a_n &\le tp2^{-n-1}n^{-3/2}\Sigma_n\{|X(\delta_1)| + \cdots + |X(\delta_1\delta_2 \cdots \delta_n)|\} \\ &= tp2^{-n-1}n^{-3/2}\{2^{n-1}(|X(0)| + |X(1)|) + 2^{n-2}(|X(00)| + |X(01)| \\ &\quad + |X(10)| + |X(11)|) \\ &\quad + \cdots + 2^{n-n}(|X(00 \cdots 0)| + \cdots + |X(11 \cdots 1)|), \end{aligned}$$

where the indices of last term appear n times,

$$= tp2^{-1}n^{-3/2}\{2^{-1}(|X(0)| + |X(1)|) + \cdots + 2^{-n}(|X(00\cdots 0)| + \cdots + |X(11\cdots 1)|)\}.$$

Similarly we can write

$$b_n = t(n+p)^{-1/2}\{2^{-n-1}\Sigma_{n+1}|X(\delta_1\delta_2\cdots\delta_{n+1})| + \cdots + 2^{-n-p}\Sigma_{n+p}|X(\delta_1\delta_2\cdots\delta_{n+p})|\}.$$

Let $Y_k = 2^{-k}\Sigma_k|X(\delta_1\delta_2\cdots\delta_k)|$ for $k = 1, 2, \cdots$; then the r.v's. $\{Y_k\}$ are independent with common mean $E(Y_k) = E|X_0| = \mu$, and variance, Var $(Y_k) = 2^{-k}$.

In view of a well known theorem for series of independent random variables, $\Sigma_k \operatorname{Var}(Y_k) < \infty \Rightarrow \Sigma_{k=1}^{n}(Y_k - \mu)$ converges a.s. (ω), hence

$$0 \le a_n \le tp2^{-1}n^{-3/2}(Y_1 + Y_2 + \cdots + Y_n - n\mu) + tp2^{-1}n^{-3/2}n\mu$$

and $0 \le b_n \le t(n+p)^{-1/2}(Y_{n+1} + \cdots + Y_{n+p} - p\mu) + t(n+p)^{-1/2}p\mu.$

Now observe that the last terms of both inequalities coverge a.s. (ω) to zero uniformly in p integer between 1 and $n^{2^{-1}-\epsilon}$. Q.e.d.

Let $\alpha(n, k)$ = number of pairs of sequences of length n, $\delta_1\delta_2\cdots\delta_n$ and $\delta_1'\delta_2'\cdots\delta_n'$, which have exactly the first k terms equal. It is easy to see that

$$\begin{aligned}\alpha(n,k) &= 2^{2n-k-1} \quad \text{if } k = 0, 1, \cdots, n-1\\ &= 2^n \quad \text{if } k = n.\end{aligned}$$

LEMMA 2. *For each t and ϵ an arbitrary positive number*

$$\psi_{[n^{1+\epsilon}]}(t, \cdot) \to e^{-t^2/2} \text{ a.s. } (\omega) \textit{ when } n \to \infty.$$

Proof. First observe that

$$E(\psi_n(t, \cdot)) = \phi^n(tn^{-1/2}) \to e^{-t^2/2} \quad \text{when } n \to \infty$$

and

$$\operatorname{Var}(\psi_n(t, \cdot)) = E|\psi_n(t, \cdot) - \phi^n(tn^{-1/2})|^2 = E|\psi_n(t, \cdot)|^2 - |\phi^n(tn^{-1/2})|^2.$$

Hence by Borel-Cantelli's lemma it is enough to show that Var $(\psi_n(t, \cdot))$ goes to zero with a speed of at least n^{-1}, because then $\psi_{[n^{1+\epsilon}]}(t, \cdot) - \phi^{n^{1+\epsilon}}(t[n^{1+\epsilon}]^{-1/2}) \to 0$ a.s. (ω) when $n \to \infty$, and this together with the first observation proves the lemma.

$$\begin{aligned}E|\psi_n(t,\cdot)|^2 &= E\{2^{-2n}\Sigma_n \exp(itn^{-1/2}S(\delta_1\delta_2\cdots\delta_n))\Sigma_n' \exp(-itn^{-1/2}S(\delta_1'\delta_2'\cdots\delta_n'))\}\\ &= E\{2^{-2n}\Sigma_n\Sigma_n' \exp itn^{-1/2}(S(\delta_1\delta_2\cdots\delta_n) - S(\delta_1'\delta_2'\cdots\delta_n'))\},\end{aligned}$$

since $\phi(tn^{-1/2}) \ne 0$ for n sufficiently large,

$$\begin{aligned}&= 2^{-2n}\Sigma_{k=0}^{n}\alpha(n,k)|\varnothing(tn^{-1/2})|^{2n-2k}\\ &= 2^{-n} + 2^{-1}|\varnothing(tn^{-1/2})|^{2n}\Sigma_{k=0}^{n-1}\{2|\varnothing(tn^{-1/2})|\}^{-2k},\end{aligned}$$

Hence

$$\operatorname{var}(\psi_n(t,\cdot)) \leq 2^{-n} + 2^{-1} \mid \phi(tn^{-1/2}) \mid^{2n} \left\{\frac{1 + 2^{-n} - 2 + \mid \varnothing(tn^{-1/2} \mid^{-2}}{1 - 2^{-1} \mid \varnothing(tn^{-1/2} \mid^{-2}}\right\}.$$

and it suffices to show that $|\phi(tn^{-1/2})|^{-2} - 1$ goes to zero with a speed of at least (const.)n^{-1}.

The Taylor expansion of $\phi(\cdot)$ gives

$$|\phi(tn^{-1/2})|^2 = 1 - t^2 n^{-1}(1 + \epsilon(n)), \text{ where } \epsilon(n) \to 0 \text{ when } n \to \infty,$$

hence

$$0 \leq |\phi(tn^{-1/2})|^{-2} - 1 = t^2 n^{-1}(1 + \epsilon(n)) |\phi(tn^{-1/2})|^{-2} \leq (\text{const.})n^{-1}.$$

COROLLARY. *The sequence* $\{\psi_n(t, \omega)\}$ *converges in mean square to the characteristic function of the normal* 0, 1 *distribution.*

LEMMA 3. *For each* t, $\psi_n(t, \cdot) \to e^{-t^2/2}$ a.s. (ω) *when* $n \to \infty$.

Proof. By Lemma 2

$$\psi_{[n^{1+\epsilon}]}(t, \cdot) \to e^{-t^2/2} \text{ a.s. } (\omega) \quad \text{when } n \to \infty,$$

so this lemma will be proven once we show that

$$\sup_{[n^{1+\epsilon}] \leq \nu \leq [(n+1)^{1+\epsilon}]} |\psi_\nu(t, \cdot) - \psi_{[n^{1+\epsilon}]}(t, \cdot)| \to 0 \text{ a.s. } (\omega)$$

when $n \to \infty$, that is, the oscillations of $\psi_\nu(t, \cdot)$ on the intervals $([n^{1+\epsilon}], [(n + 1)^{1+\epsilon}])$ tend to zero a.s. (ω) when $n \to \infty$; this follows from the easily verified fact

$$([n^{1+\epsilon}], [(n + 1)^{1+\epsilon}]) \subset ([n^{1+\epsilon}], [n^{1+\epsilon}] + [n^{1+\epsilon}]^{2^{-1}-\epsilon})$$

for $\epsilon > 0$ small, n large enough and Lemma 1.

LEMMA 4. $\mathfrak{L}\{t : \psi_n(t, \omega) \nrightarrow e^{-t^2/2}\} = 0$ *for* a.s. (ω).

Proof. By Lemma 3 we have for each t

$$P\{\omega : \psi_n(t, \omega) \nrightarrow e^{-t^2/2}\} = 0.$$

Let $A = \{(t, \omega) : \psi_n(t, \omega) \nrightarrow e^{-t^2/2}\}$, and A_t, A_w be the t and ω sections of A. Then by Fubini's theorem

$$0 = \int_R \int_\Omega I_{A_t} \, dP \, d\mathfrak{L} = \int_\Omega \int_R I_{A_\omega} \, d\mathfrak{L} \, dP,$$

hence a.s. (ω) $\mathfrak{L}(A_w) = 0$.

UNIVERSITY OF MONTREAL and
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

REFERENCES

[1] T. E. HARRIS, The theory of branching processes, Springer-Verlag (1963) p. 75.
[2] P. E. NEY, *The limit distribution of a binary cascade process*, J. Math. Anal. Appl. **10** (1965), 30–6.
[3] ———, *The convergence of a random distribution function associated with branching process*, J. Math. Anal. Appl. **12** (1965), 316–27.