DIFFERENTIAL STRUCTURES AND TANGENT BUNDLES ON BANACH MANIFOLDS(*)

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1. Introduction and Statement of Results

A separable metrizable manifold modeled on an infinite dimensional separable Banach space B will be called a B-manifold.

A Banach space B such that $B \times B$ is isomorphic with B will be called a stable Banach space and a B-bundle $\xi : E \to X$ over any space X will be called stable if the Whitney sum $\xi \oplus 1_x$ is equivalent to ξ. Here 1_x is the trivial B-bundle over X.

Note that $\xi \oplus 1_x$ is always stable and that if one of two equivalent B-bundles is stable then both are stable.

Henderson [6] has shown that every topological B-manifold is homeomorphic to an open subset of B, for which reason it has a differentiable structure with trivial tangent bundle. This particular fact led us to the following:

Problem. Determine the B-bundles over a B-manifold M which are tangent bundles of M for some smooth structure.

Note that if the general linear group of B is contractible the problem is trivial.

We found that for stable Banach spaces B with a C^∞-norm essentially all bundles over a B-manifold M with fiber B are tangent bundles. To be more explicit: let $\text{Vect}_B(X)$ be the set of equivalence classes of B-bundles over X; let $\text{St}_B(X)$ be the subset of those elements of $\text{Vect}_B(X)$ which contain stable B-bundles and let $\text{Tan}(M)$ be the subset of $\text{Vect}_B(M)$ consisting of classes which contain a B-bundle which is the tangent bundle of M for some smooth structure. We obtain the following results:

Theorem 1.1 *Let B be a stable Banach space with a C^∞-norm and let M be a topological B-manifold. Then*

$$\text{St}_B(M) \subset \text{Tan}(M).$$

In other words, any stable B-bundle over M is equivalent to the tangent bundle of M for some smooth structure.

In the case of the stable Banach space $B = c_0 \times \ell^2$ for which the general linear group is not contractible, as shown by Douady [5], we prove

Theorem 1.2 *If $B = c_0 \times \ell^2$ and X is a paracompact space then* $\text{St}_B(X) = \text{Vect}_B(X)$.

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These theorems solve the problem completely for the case $B = c_0 \times \ell^2$, namely, every $c_0 \times \ell^2$-bundle over a $c_0 \times \ell^2$-manifold M is equivalent to the tangent bundle of M for some smooth structure. It would be interesting to know whether Theorem 1.2 holds in general. Also, when do we have $\mathrm{St}_B(M) = \mathrm{Tan}(M)$?

2. Proof of Theorem 1.1

To prove Theorem 1.1 we first need some lemmas.

Let $K(\mathrm{Vect}_B(X))$ denote the Grothendieck group associated to $\mathrm{Vect}_B(X)$.

LEMMA 2.1 *Let X be a paracompact space and B a stable Banach space. Then, $\mathrm{St}_B(X)$ is a group under the Whitney sum. Moreover,*

$$K(\mathrm{Vect}_B(X)) = \mathrm{St}_B(X)$$

Proof. Given a B-bundle $\xi : E \to X$, the existence of $\xi' : E' \to X$ such that $\xi \oplus \xi' = 1_X$ follows using standard techniques and the fact that B is stable. It follows from the definition of $\mathrm{St}_B(X)$ that 1_X is the identity element. The last part follows from the definition of K.

LEMMA 2.2. *Let B be a stable Banach space with a C^∞-norm. Let M be a C^∞-B-manifold. Then, given any class $[\pi] \in \mathrm{Vect}_B(M)$, there exists π', a C^∞-B-bundle over M, such that $[\pi'] = [\pi]$.*

Proof. Let G_B be the set of all closed subspaces of B, isomorphic to B, and admitting a closed complement isomorphic with B. On G_B consider the following metric:

$$d(B', B'') = \theta(B', B'') + \theta(B'', B')$$

where

$$\theta(B', B'') = \sup_{\substack{x \in B' \\ \|x\| \le 1}} \left(\inf_{\substack{y \in B'' \\ \|y\| \le 1}} \|x - y\| \right)$$

for $B', B'' \in G_B$.

Douady [4] has proved that G_B admits an analytic atlas.

Let $p : G_B \times B \to G_B$ be the projection on the first factor, and $E = \{(X, y) \in G_B \times B \mid y \in X\}$, then $p \mid E : E \to G_B$ is a C^∞-B-bundle.

Since π has a complementary B-bundle, the existence of an exact sequence $0 \to \pi \xrightarrow{h} M \times B$ follows. Let $p : M \times B \to B$ be the projection on the second factor. Define $f : M \to G_B$ by $f(x) = ph(\pi^{-1}(x))$. It is continuous and the pull back of f is equivalent with π.

Since B has a C^∞-norm there exist C^∞-partitions of 1. By the usual argument with the C^∞-partitions of 1 we can find a C^∞-function $g : M \to G_B$ homotopic to f. We let π' be the pullback of g.

Proof of theorem 1.1 Consider in M a parallelizable C^∞-structure and apply Lemma 2.2 to get a C^∞-bundle $\pi' : E \to M$ in the class of $[\pi]$.

The bundle atlas introduces in E a C^∞-B-manifold structure.

Since E has the same homotopy type as M (the zero section is a deformation retract of E and is homeomorphic with M), we have that E is homeomorphic with M (see Burghelea and Kuiper [2]). We claim this C^∞-structure on E is the one we are looking for. To check this, it is enough to look at the zero section of π'. The tangent bundle to E restricted to the zero section is $\pi' \oplus 1_B$ where π' is given by the derivative in the direction of the fiber and 1_B comes from the derivative in the direction of the manifold M with respect to the parallelizable structure we chose. By Lemma 2.1 the theorem follows.

3. The Case $B = c_0 \times \ell^2$

Let $GL(B)$ denote the general linear group of B. If $GL(B)$ is contractible to a point, as in the case of ℓ^2 or c_0 (see Kuiper [7] and Arlt [1], we have that

$$\mathrm{Vect}_B(X) = \mathrm{St}_B(X) = \{1_x\}$$

for all X paracompact. Therefore, the theorem is trivial in this case.

Nevertheless, this is not the general case. Douady [5] has shown that $GL(c_0 \times \ell^2)$ has the same homotopy type as $Z \times BO$, hence

$$\mathrm{Vect}_{c_0 \times \ell^2} \neq \{1_X\}$$

in general.

Since it was shown by Kuiper that c_0 has a C^∞-norm (see Boric and Framton [3]) it follows that $c_0 \times \ell^2$ has a C^∞-norm. It is clear that $c_0 \times \ell^2$ is stable.

Proof of theorem 1.2 Let $G = \{T \in GL(B \times B) \mid T(x, y) = (T'(x), y), T' \in GL(B)\}$. Since both G and $GL(B \times B)$ are isomorphic to $GL(B)$ we have an inclusion of the group $GL(B)$ in itself, which we must now prove to be a homotopy equivalence. The theorem will then follow from the results of Milnor [8].

Since $GL(B)$ has the homotopy type of a CW-complex (Milnor [9]) weak homotopy equivalence implies homotopy equivalence (Milnor, *op. cit.*).

Observe that if

$$T: \ell^2 \times \ell^2 \to \ell^2$$

is an isomorphism, and

$$f: X \to \mathfrak{F}(\ell^2)$$

is a continuous map from a compact space X into the space of Fredholm operators in ℓ^2 then the maps

$$(f, id): X \to \mathfrak{F}(\ell^2 \times \ell^2)$$

given by $(f, id)(x) = (f(x), x): \ell^2 \times \ell^2 \to \ell^2 \times \ell^2$ and $T^*f: X \to \mathfrak{F}(\ell^2 \times \ell^2)$ given by

$$T^*f(x) = Tf(x)T^{-1}$$

have the same Jänich Index.

It now follows from Douady [5] that the inclusion map $G \subset GL(B \times B)$ induces isomorphisms of the homotopy groups, *i.e.* is a weak homotopy equivalence. This concludes the proof of Theorem 1.2.

Observe that, in particular, $S^1 \times (c_0 \times \ell^2)$ admits countably many distinct differentiable structures with distinct tangent bundles.

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