MAPS BETWEEN HURWITZ SPACES

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Introduction

Consider the Hurwitz Spaces $H(n, s)$ i.e. the complex manifolds whose points are the isomorphism classes of n-sheeted branched covers of the projective line $\mathbb{P}^1$ with s branch points. In this paper we study some global maps among Hurwitz spaces. In proposition 1 we prove that some group homomorphisms, $S_n \to S_m$ between symmetric groups, induce in a natural way maps among certain connected components of $H(n, s)$ and $H(m, s)$. We also show that such maps commute with the action of $PGL(1)$ (proposition 2).

Let $H^{n,s}$ be the connected component of $H(n, s)$ consisting of simple covers. Among our maps we have an injective map $B: H^{4,s} \to H(6, s)$ and a surjective map $K: H^{4,s} \to H^{3,s}$.

In theorem 1 we prove that every element of $B(H^{4,s})$ admits a fixed point free involution and we also prove that the map which assigns to each cover its quotient by the involution is precisely the map $K \cdot B^{-1}: B(H^{4,s}) \to H^{3,s}$.

Let $\mathfrak{M}_g(\mathfrak{M}_g^h)$ be the variety of moduli of curves (hyperelliptic) of genus g. In theorem 2 we prove that we have dense maps:

$$\begin{array}{ccc} H^{g,4g-2} & & \\ \downarrow & \searrow & \\ \mathfrak{M}_g & & \mathfrak{M}^h_{2g-2} \end{array} \qquad\qquad \begin{array}{ccc} H^{g+1,4g} & & \\ \downarrow & \searrow & \\ \mathfrak{M}_g & & \mathfrak{M}^h_{2g-1} \end{array}$$

the fibers of the diagonal maps being of dimension equal to dim $PGL(1)$.

1. Some homomorphisms among symmetric groups

Let $b_{\binom{n}{r}}: S_n \to S_{\binom{n}{r}}$ be the faithful representation of S_n, given by its action on the left cosets of the subgroup which leaves fixed a set of r symbols $\{j_1, \cdots, j_r\} \subset \{1, \cdots, n\}$ $(r \leq n/2)$. The image of an r-transitive subgroup under this monomorphism is a transitive subgroup, and if one has the transposition (ij) on S_n, its image on $S_{\binom{n}{r}}$ will be the product of $\binom{n-2}{r-1}$ disjoint transpositions:

$$\prod_{i_k \neq i,j}^{i_k \in \{1, \cdots, n\}} (\{i, i_2, \cdots, i_r\}\{j, i_2, \cdots, i_r\}).$$

Consider also $\kappa: S_4 \to S_4/\kappa_4 \cong S_3$ where κ_4 is the Klein group. The image of a transposition is also a transposition, but the image of a transitive subgroup is not necessarily transitive.

Let $F = \{1, t, t', tt'\} \subset S_4$, where t and t' are disjoint transpositions. F is a subgroup of Index 2 of $\kappa^{-1}(\kappa(F))$.

2. Maps between Hurwitz spaces

We use here some classical definitions and results as they appear in chapter 1 of Fulton [1]. Let $\mathbb{P}^1$ be the complex projective line, and let Σ^s be the complex manifold consisting of the unordered s-tuples of different points of $\mathbb{P}^1$. Let $H(n, s)$ be the set of isomorphisms classes of n-sheeted branched covers of $\mathbb{P}^1$ from a compact Riemann surface, with s-branch points. There is a topology on $H(n, s)$ such that the function $\delta: H(n, s) \to \Sigma^s$ which associates to each branched cover its branch locus, becomes an analytic cover map. Let $H^{n,s}$ be the subset of $H(n, s)$ corresponding to the simply ramified covers.

Given $A = \{a_1, \cdots, a_s\} \in \Sigma^s$, let $\sigma_1, \cdots, \sigma_s$ be closed curves "around" $a_1, \cdots, a_s$ respectively, such that $\sigma_1, \cdots, \sigma_s$ are generators of $\pi_1(\mathbb{P}^1 - A, x)$ with the one relation $\sigma_1 \cdots \sigma_s = 1$. Let $\mathrm{Hom}(\pi_1(\mathbb{P}^1 - A, x), S_n)$ denote the set of equivalence classes $\bar{f}$ of homomorphisms $f: \pi_1(\mathbb{P}^1 - A, x) \to S_n$, whose image is a transitive subgroup of S_n. Let $\mathrm{Hom}^t(\pi_1(\mathbb{P}^1 - A, x), S_n)$ be the subset corresponding to those homomorphisms f such that $f(\sigma_i)$ is a transposition for $i = 1, \cdots, s$.

The fibers $H(n, A) = \delta^{-1}(A)$ are in natural one to one correspondence with $\mathrm{Hom}(\pi_1(\mathbb{P}^1 - A, x), S_n)$, under such correspondence $H^{n,A}$ corresponds to $\mathrm{Hom}^t(\pi_1(\mathbb{P}^1 - A, x), S_n)$.

From now on we will denote by $\bar{f}$ either the class of the cover $f_c: X \to \mathbb{P}^1$ or the corresponding class of the homomorphism $f: \pi_1(\mathbb{P}^1 - A, x) \to S_n$, where $A = \delta(f_c)$.

Observe that from the Riemann-Hurwitz formula it follows that $H^{n,s} \neq \emptyset$ if $s - 2n \geq 0$ (Gunning [3]).

Now we recall the topological structure of the map $\delta: H(n, s) \to \Sigma^s$. Let $N(U_1, \cdots, U_s)$ be the subset of Σ^s consisting of the s-tuples of points, having one point on each U_i, where the U_i are disjoint open disks in $\mathbb{P}^1$. So for any $A, A' \in N(U_1, \cdots, U_s)$, since $\mathbb{P}^1 - U$ is a deformation retract of $\mathbb{P}^1 - A$ and $\mathbb{P}^1 - A'$ where $U = \bigcup_{i=1}^n U_i$, we have isomorphisms $\varphi_{A,A'}: \pi_1(\mathbb{P}^1 - A, x) \xrightarrow{\cong} \pi_1(\mathbb{P}^1 - U, x) \xrightarrow{\cong} \pi_1(\mathbb{P}^1 - A', x)$; and such ismorphisms do not depend on $U_1, \cdots, U_s$.

Hence, if $\bar{f} \in H(n, s)$ is such that $\delta(\bar{f}) \in N(U_1, \cdots, U_s)$, the neighborhood above $N(U_1, \cdots, U_s)$ which contains $\bar{f}$, can be identified with:
$N(U_1, \cdots, U_s)_{\bar{f}} = \{\overline{f \circ \varphi_{A,A'}} \in \mathrm{Hom}(\pi_1(\mathbb{P}^1 - A', x), S_n) \mid A' \in N(U_1, \cdots, U_s)\}$

PROPOSITION 1. *The Σ^s-maps: $B_{\binom{n}{r}}: H^{n,s} \to H(\binom{n}{r}, s)$, defined as: $B_{\binom{n}{r}}(\bar{f}) = \overline{b\binom{n}{r} \circ f}$ $K: H^{4,s} \to H^{3,s}$, defined as $K(\bar{f}) = \overline{\kappa \circ f}$, are analytic cover maps.*

Proof. The maps are clearly well defined and they are analytic since:

$$B_{\binom{n}{r}}(N(U_1, \cdots, U_s)_{\bar{f}}) = N(U_1, \cdots, U_s)_{\overline{b\binom{n}{r} \cdot f}}$$

and

$$K(N(U_1, \cdots, U_s)_{\bar{f}}) = N(U_1, \cdots, U_s)_{\overline{\kappa \circ f}}.$$

Remark 1. $\delta: H^{n,s} \to \Sigma^s$ can be considered as the map induced by the natural epimorphism $S_n \to S_2$.

Remark 2. Our result is local: Let $\bar{f} \in H(n, s)$. If there exists an homomorphism

$\lambda: S_n \to S_m$ such that $\lambda \circ f \in H(m, s)$, then there exists a neighborhood $N(U_1, \cdots, U_s)$ of $\delta(\bar{f})$ and an analytic cover map:

$$\Lambda: N(U_1, \cdots, U_s)_{\bar{f}} \to N(U_1, \cdots, U_s)_{\overline{\lambda \circ f}}.$$

The action of the projective group $PGL(1)$ on $\mathbf{P}^1$, induces an action on Σ^s action that can be lifted to $H(n, s)$.

$$\begin{aligned} PGL(1) \times H(n, s) &\to H(n, s) \\ (\sigma, \bar{f}) &\to \overline{f \circ (\sigma^{-1})^*} \end{aligned}$$

where $\sigma^*: \pi_1(\mathbf{P}^1 - A, x) \to \pi_1(\mathbf{P}^1 - A^\sigma, h^\sigma)$ is the one induced by σ. Since $B_{\binom{n}{r}} \overline{(f \circ (\sigma^{-1})^*)} = \overline{b_{\binom{n}{r}} \circ f \circ (\sigma^{-1})^*}$ and $K\overline{(f \circ (\sigma^{-1})^*)} = \overline{\kappa \circ f \circ (\sigma^{-1})^*}$ we have:

PROPOSITION 2. *The maps K and $B_{\binom{n}{r}}$ commute with the action of $PGL(1)$ on $H^{n,s}$.*

PROPOSITION 3.

a) $B_{\binom{n}{r}}: H^{n,s} \to H_{\binom{n}{r}}, s)$ *is an injective map for* $n \neq 6$.

b) $K: H^{4,s} = H^{3,s}$ *is surjective and of degree* $2^{s-4} - 1$.

Proof. Part a) follows from the fact that the $B_{\binom{n}{r}}$ are monomorphisms, and from the fact that S_n does not have any exterior automophisms for $n \neq 6$ (Scott [4]).

To prove c) notice that $H^{n,A}$ can be identified with the set $S(n, s)$ of equivalence classes $[t_1, \cdots, t_s]$ of s-tuples of transpositions, via $\bar{f} \to [f(\sigma_1), \cdots, f(\sigma_s)]$; moreover given $\bar{f} \in H^{n,s}$, one can always choose the basis $\sigma_1, \cdots, \sigma_s$ such that $\bar{f} \leftrightarrow [(12), (12), (13), (13), \cdots, (1n), \cdots, (1n)]$ where $(1n)$ appears $s - 2n + 4$ times (Fulton [1]).

So consider $T = [(12), (12), (13), \cdots, (13)] \in S(3, s)$, and construct all possible $[t_1, \cdots, t_s] \in S(4, s)$ such that $\kappa^{\#}([t_1, \cdots, t_s]) = [\kappa(t_1), \cdots, \kappa(t_s)] = T$. To do this consider the following explicit form of κ:

$$\begin{aligned} \kappa(12) &= \kappa(34) = (12) \\ \kappa(13) &= \kappa(24) = (13) \\ \kappa(23) &= \kappa(14) = (23) \end{aligned}$$

and then a short computation shows that degree of $\kappa^{\#}$ is precisely $2^{s-4} - 1$.

3. Some geometric consequences

The relations between the genera are of interest. If X is a compact Riemann surface g_X will denote its genus.

3.1

Consider branched covers $f_c: X \to \mathbf{P}^1$ and $g_c: Y \to \mathbf{P}^1$ such that $\bar{f} \in H^{4,s}$, $\bar{g} \in H^{3,s}$ and $K(\bar{f}) = \bar{g}$. Both covers have the same branch locus and both are simple covers; hence they have the same ramification index. So by the Riemann-Hurwitz formula we have: $2(g_X + 4 - 1) = s = 2(g_Y + 3 - 1)$. Hence $g_Y = g_X + 1$.

3.2

Let $f_c: X \to \mathcal{P}^1$ and $h_c: Z \to \mathcal{P}^1$ be branched covers such that $\bar{f} \in H^{n,s}$, $\bar{h} \in H(\binom{n}{r}, s)$ and $B_{\binom{n}{r}}(\bar{f}) = \bar{h}$. Since f_c is a simple cover, we have $s = 2(g_X + n - 1)$. Let $\delta(f_c) = A = \{a_1, \cdots, a_s\}$ and $\sigma_1, \cdots, \sigma_s$ as before. Hence we can assume that for $i = 1, \cdots, s$ we have: $h(\sigma_i) = b_{\binom{n}{r}}(f(\sigma_i))$, but $f(\sigma_i)$ is a transposition; thus $h(\sigma_i)$ is the product of $\binom{n-2}{r-1}$ disjoint transpositions, i.e. h_c has $\binom{n-2}{r-1}$ ramification points above each a_i, each with ramification index equal to 2. So the Riemann-Hurwitz formula gives us: $\binom{n-2}{r-1}s = 2(g_Z + \binom{n}{r} - 1)$. Finally, we have: $g_Z = \binom{n-2}{r-1}(n + g_X - 1) - \binom{n}{r} + 1$.

3.3

Consider $f_c: X \to \mathcal{P}^1$ and $g_c: Y \to \mathcal{P}^1$ such that $\bar{f} \in H^{n,s}$, $\bar{g} \in H^{2,s}$ and $\delta(\bar{f}) = \bar{g}$. It follows in the same way as in 3.1 that: $g_Y = g_X + n - 2$.

We now go into the relation between the maps K and $B = B\binom{4}{2}$. Let $f_c: X \to \mathcal{P}^1$ be a branched cover such that $\bar{f} \in H^{4,s}$ and as usual let $f: \pi_1 \to S_4$ be a corresponding homomorphism, where $\pi_1 = \pi_1(\mathcal{P}^1 - A, x)$ and $A = \delta(\bar{f})$. Let $g_c: Y \to \mathcal{P}^1$ be the analytic completion of the topological cover of $\mathcal{P}^1 - A$, corresponding to the subgroup $f^{-1}(\kappa(F)) \subset \pi_1$, where F is as in 2, let $h_c: Z \to S_6$ be the analytic completion of the topological cover corresponding to the subgroup

$$(b \circ f)^{-1}(b(F)) = f^{-1}(F).$$

Then we have $B(\bar{f}) = \bar{h}$ and $K(\bar{f}) = \bar{g}$; moreover since $[f^{-1}(\kappa(F)) : f^{-1}(F)] = [\kappa^{-1}(\kappa(F)) : F] = 2$, there exists an analytic map $Z \to Y$ of degree 2. From 3.1 and 3.2, it follows that $g_Z = 2g_Y + 1$, which is the Riemann-Hurwitz formula in the case that the ramification index of such map is zero, therefore our map is unramified.

Fix $g_c: Y \to \mathcal{P}^1$ as before; then, since B is an injective map, we get $2^{s-3} = 1$ non isomorphic unramified double covers, $Z \to Y$. In fact we exhaust all such covers. To see this let J_Y^2 be the group of 2-division points of the Jacobian variety of Y, so to each non trivial element of J_Y^2, there corresponds an unramified double cover of Y (Gunning [3]). Card. $J_Y^2 = 2^{2g_Y}$. Hence, since $g_Y = s_2 - 2$, it follows that Card. $J_Y^2 = 2^{s-4} - 1$. So we have proved:

THEOREM 1. $\Phi = K \circ B^{-1} : B(H^{4,s}) \to H^{3,s}$ *is an analytic cover map of degree* $2^{s-4} - 1$. *Morover, given* $g_c: Y \to \mathcal{P}^1$ *such that* $\bar{g} \in H^{3,s}$, $\Phi^{-1}(\bar{g})$ *can be identified in a natural way with the set of the unramified double covers of* Y.

3.4

The Hurwitz spaces are a special case of Hurwitz schemes (Fulton [1]). Hence by the Grauert comparation Theorem (Grothendieck [2]), we know that all our maps are algebraic. Moreover, the Hurwitz schemes carry a family of covers (universal if $n \geq 2$). So if $\mathfrak{M}_g(\mathfrak{M}_g^h)$ is the variety of moduli of irreducible, nonsingular (hyperelliptic) algebraic curves defined over $\mathcal{C}$, one has natural morphisms

$$P_{n,s} : H^{n,s} \to \mathfrak{M}_g; \qquad s = 2(n + g - 1)$$

In particular we have:

$$\begin{array}{ccc} & H^{4,s} \longrightarrow H^{3,s} & \\ P_{4,s} & \downarrow \qquad\qquad \downarrow & P_{3,s} \\ & \mathfrak{M}_g \qquad\quad \mathfrak{M}_{g-1} & \end{array}$$

which gives a relation between curves of genus $g + 1$ which carry a g_3^1, and the curves of genus g which carry a g_4^1. (g_n^r = complete linear system of degree n and dimension r).

We finally have a result, which tells us that any hyperelliptic curve of genus g_h can be constructed from a curve of genus $g_h/2 + 1$ and a g_g^1 on such curve if g_h is even, and from a curve of genus $g_h + 1/2$ and a g_{g+1}^1 if g_h is odd.

THEOREM 2. *One has the following dense maps*:

$$\begin{array}{ccc} H^{g,4g-2} & & \\ \downarrow & \searrow^{P_{g,4g-2}} & \\ \mathfrak{M}_g & & \mathfrak{M}^h_{2g-2} \end{array} \qquad \begin{array}{ccc} H^{g+1,4g} & & \\ \downarrow & \searrow^{P_{g+1,4g}} & \\ \mathfrak{M}_g & & \mathfrak{M}^h_{2g-1} \end{array}$$

for $g \geq 2$.

Proof. The diagonal maps are clearly surjective. Since our varieties are irreducible, to show that our maps are dense it is sufficient to show that their fibers have the right dimension.

Let C be an irreducible and non singular curve of genus g. Let $p_0 \in C$, and let $\varphi_n: C^{(n)} \to J_C$; $D \to [D - np_0]$ be the usual morphisms from the symmetric products of C into its Jacobian variety J_c. Let $G_g^1 = \{a \in J_C \mid \dim \varphi_n^{-1}(a) = 1\}$. For C general one can always find two independent differentials which vanish at those points. From this using Riemann-Roch it follows that $\dim G_g^1 = g - 2$.

So the dimension of the general fiber of $P_{g,4g-2}$ is: $\dim G_g^1 + \dim PGL(1) = g - 2 + 3 = g + 1$ which is the right one since: $\dim H^{g,4g-2} - \dim \mathfrak{M}_g = 4g - 2 - (3g - 3)$.

The general fiber of φ_{g-1} is of dimension 1, hence $\dim G_{g-1}^1 = g$. So the dimension of the general fiber of $P_{g,4g}$ is: $\dim G_{g+1}^1 - \dim PGL(1) = g - 3$, which again is the right one.

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