FUNCTIONAL AND ORDINARY DIFFERENTIAL EQUATIONS

BY

JAVIER GONZÁLEZ G., CARLOS IMAZ AND ZDENEK VOREL

Introduction

We continue our attempts to get a unified theory for both ordinary differential equations and functional differential equations of retarded type. Given a retarded functional differential equation we construct an ordinary differential equation in L_p, $p \geq 1$, such that solutions (in the sense of Caratheodory) of the first equation, with Lebesgue integrable initial function, correspond to solutions of the second equation. The construction is the same as in [1], but the condition of Riemann integrability in [1] is replaced by Lebesgue integrability. The proofs, as one should expect, differ considerably. The functional equations considered are as in [2] so that, in particular, equations studied in [3] are included as a special case.

The idea of associating a generalized (in the sense of [4]) ordinary differential equation to functional retarded equations was first pointed out by Kurzweil and studied in [5] and [6]. The step from generalized to classical ordinary equations is made possible by working in L_p instead of spaces of continuous functions.

Somewhat similar results have been obtained in [7] in the case of boundary value problems for linear functional equations.

Notation

Let a and h be positive numbers, $p \in [1, \infty)$ and let $L_p([-h, a], \mathbf{R}^n)$ denote, as usual, the set of equivalence classes of functions x from $[-h, a]$ into $\mathbf{R}^n$ such that $|x|^p$ is Lebesgue integrable. With $A \subset L_p([-h, a], \mathbf{R}^n)$ we denote the set of equivalence classes $[x]$ such that the class of restrictions $[x]_{[0,a]}$ of $[x]$ to the interval $[0, a]$ contains an absolutely continuous function. As usual we shall not distinguish between a class $[x]$ or its elements x unless there is danger of confusion. For example, given a function $x:[-h, a] \to \mathbf{R}^n$, the symbol x_t denotes the function from $[-h, a] \to \mathbf{R}^n$ defined by $x_t(s) = x(s)$ if $-h \leq s < t$ and $x_t(s) = x(t)$ if $t \leq s \leq a$; that is x_t is the truncation of x at the point t. It is now clear that no confusion arises between $[x_t]$ and x_t if $[x] \in A$, $t \geq 0$, and we are always taking a representative continuous in $[0, a]$.

The theorems

Let $B \subset A \subset L_p([-h, a], \mathbf{R}^n)$ be a subset with the property that $x \in B \to x_t \in B$ for every $t \in [0, a]$. Consider now the functional differential equation

$$(1) \qquad \frac{dx(t)}{dt} = f(x_t, t),$$

where $f:B \times [0, a] \to \mathbf{R}^n$. A function $x:[-h, a] \to \mathbf{R}^n$ is called a solution to (1) with initial condition $x_0 \in B$ if $x \in B$ and

$$(2)\qquad \begin{aligned} x(t) &= x(0) + \int_0^t f(x_s, s)\,ds \quad \text{for} \quad 0 \le t \le a \\ x(t) &= x_0(t) \quad \text{for} \quad -h \le t < 0. \end{aligned}$$

Observe that this type of functional equations are more general than those considered, for example, in [3].

Our aim is to construct an ordinary differential equation in $L_p([-h, a], \mathbf{R}^n)$ equivalent to (1) under the assumption that $f(x., \cdot) \in L_p([0, a], \mathbf{R}^n)$ for all fixed $x \in B$. In order to do this we define a function $G: B \times [0, a] \to L_p([-h, a], \mathbf{R}^n)$ as follows:

$$(3)\qquad G(x, t)(\tau) = \begin{cases} 0 & \text{for} \quad -h \le \tau < t \\ f(x_t, t) & \text{for} \quad t \le \tau \le a. \end{cases}$$

In other words given $x \in B$, $t \in [0, a]$, $G(x, t)$ is a representative of the class in $L_p([-h, a], \mathbf{R}^n)$ that contains the step function with values 0 on $[-h, t)$ and $f(x_t, t)$ on $[t, a]$.

We now consider the ordinary differential equation

$$(4)\qquad \frac{dy}{dt} = G(y, t).$$

Let $[y_0]$ be a class in B that contains a representative y_0 whose restriction to $[0, a]$ is constant, that is, $y_0(s) = y_0(0)$ for $s \in [0, a]$. Then a solution of (4) with initial condition y_0 is a function $y: [0, a] \to B$ such that

$$y(t) = y_0 + \int_0^t G[y(s), s]\,ds, \quad \text{for} \quad 0 \le t \le a.$$

Here the integral is taken in the sense of Bochner [8].

We can now summarize our results in the following two theorems.

THEOREM 1. *Let $f(x., \cdot) \in L_p([0, a], \mathbf{R}^n)$ for every $x \in B$. Let x be a solution of* (1) *with initial condition $x_0 \in B$, the restriction of x_0 of* $[0, a]$ *being a constant function. Then the function $y: [0, a] \to B$ defined by $y(t) = x_t$ for $t \in [0, a]$ is a solution of* (4) *in* $[0, a]$ *with initial condition $y(0) = x_0$.*

THEOREM 2. *Let $f(x., \cdot) \in L_p([0, a], \mathbf{R}^n)$ for every $x \in B$. Let y be a solution of* (4) *in* $[0, a]$ *with initial condition $y(0) = y_0$. Then there exists $x \in B$ such that $y(t) = x_t$ for $t \in [0, a]$ and such x is a solution of* (1) *in* $[0, a]$ *with initial condition $x_0 = y_0$.*

Note. Since the function $y(t)$ of Theorem 2 is a class in B, We can select a representative that is continuous in $[0, a]$. Therefore, the existence of $x \in B$ such that $x_t = y(t)$ implies, for the continuous representative, that $y(t)(s) = y(t)(t)$ in $0 \le t \le s \le a$ and $y(s)(t) = y(t)(t) = x(t)$ in $0 \le t \le s \le a$. This, of course, is due to the special way of constructing G from f.

Theorems 1 and 2 will be proved by means of the following 5 Lemmas.

Lemmas

LEMMA 1. *Let X and Y be Banach spaces and let $f:[0, a] \to X$ and $T:[0, a] \to \mathcal{L}(X, Y)$ be functions such that, $f \in L_p([0, a], X)$ and T is a continuous function with values in the Banach space of bounded linear operators from X to Y, then g: $[0, a] \to Y$ defined by $g(s) = T(s)(f(s))$ belongs to $L_p([0, a], Y)$.*

Proof. By hypothesis there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions such that $f_n:[0, a] \to X, f_n \to f$ a.e. and $f_n \to f$ in L_p.

Now, for each n we define a function $g_n:[0, a] \to Y$ by the formula $g_n(s) = T(s)(f_n(s))$; then g_n is continuous, moreover $g_n \to g$ a.e.

Now we will show that $\{g_n\}$ is a Cauchy sequence in $L_p([0, a], Y)$.

As T is bounded

$$\int_0^a |g_n(s) - g_m(s)|^p ds \leq \int_0^a |T(s)|^p |f_n(s) - f_m(s)|^p ds$$
$$\leq k \int_0^a |f_n(s) - f_m(s)|^p ds$$

converges to zero if $n, m \to \infty$. As $L_p[[0, a], Y]$ is complete, $\{g_n\}$ converges to some $\tilde{g} \in L_p[[0, a], Y]$, but since $\{g_n\}$ converges to g a.e., $g = \tilde{g}$ a.e.

COROLLARY. *Let $x \in B$ be fixed and suppose $f(x., \cdot) \in L_p([0, a], \mathbf{R}^n)$. Let $G(x, \cdot)$ be given by (3). Then*

$$g(x, \cdot) \in \sum L_p([0, a], L_p([-h, a], \mathbf{R}^n)).$$

Proof. Define the family $T(t)$, $t \in [0, a]$ of operators $T(t):\mathbf{R}^n \to L_p([-h, a]$ $\mathbf{R}^n)$ by

$$[T(t)y](\tau) \begin{cases} 0 & \text{for } \quad -h \leq \tau < t \\ y & \text{for } \quad t \leq \tau \leq a \end{cases}$$

for all $y \in \mathbf{R}^n$.

LEMMA 2. *Let $H \in L_p([0, a], L_p([-h, a], \mathbf{R}^n))$ and let $I = [\tau_1, \tau_2] \subset [-h, a]$. Then for any $t \in [0, a]$ we have that $\int_0^t [H(s)]_I ds$ exists and is equal to $[\int_0^t H(s)ds]_I$ where $[\quad]_I$ indicates restriction to I.*

Proof. Consider now the operator T from $L_p([-h, a], \mathbf{R}^n)$ into $L_p[I, R^n]$, defined by $T[x] = [x]_I$. Obviously this is a linear bounded operator and the lemma follows from a well known result [8].

LEMMA 3. *Let G be defined by (3) and let $y:[0, a] \to B$ be such that $\int_0^a G[y(s), s]ds$ exists. Let $t \in [0, a]$, $\tau \in [t, a]$ and let I be a subinterval of $[0, t]$. Then*

(5) $[\int_0^t G[y(s), s]ds]_I = [\int_0^\tau G[y(s), s]ds]_I$, *and the restriction* $[\int_0^t G[y(s), s]ds]_{[t,a]}$ *is a constant function.*

Proof. By Lemma 2 $[\int_t^\tau G[y(s), s]ds]_I \geq \int_t^\tau [G[y(s), s]_I ds$. By (3) $G[y(s), s](\tau) = 0$ if $s > \tau$, which proves (5). On the other hand, since the restriction $[G[y(s), s]]_{[t,a]}$ is a constant function with values in $\mathbf{R}^n$ for $0 \leq s \leq t$, so will be its integral and the last part of the Lemma follows from Lemma 2.

LEMMA 4. *Let* $[g] \in L_p([-h, a], \mathbf{R})$ *and let* α *be a continuous function from* $[-h, a]$ *into* $\mathbf{R}$. *Then* α *is an element of the class* $[g]$ *if and only if for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that for some* $g \in [g]$, *and for every* $t_0 \in [-h, a]$ *we have* $| g(t) - \alpha(t_0) | < \epsilon$ *for almost every* $t \in [t_0 - \delta \cdot t_0 + \delta] \cap [-h, a]$.

Proof. If α belongs to [g] the result follows from the uniform continuity of α.

On the other hand assume the relation satisfied for some function g representing the class. Consider the rationals in $[-h, a]$ numbered in some way $\{r_i\}$, $i = 1, 2, 3, \cdots$; and consider a sequence $\{\epsilon_k\}$ of positive numbers converging to zero as $k \to \infty$. From the hypothesis, to each ϵ_k there corresponds a δ_k and to each rational r_i a set $E_i^k \subset [-h, a]$ such that the Lebesque measure of E_i^k is zero and

$$(6) \qquad | g(t) - \alpha(r_i) | < \epsilon_k$$

for all $t \in [r_i - \delta_k, r_i + \delta_k] \cap [-h, a] - E_i^k$.

Denote $E = \bigcup_{i,k}^{\infty} E_i^k$, which has measure zero. To finish the proof it is enough to show that $g = \alpha$ on the complement of E. Assume there is a $t \in [-h, a] - E$ such that $| g(t) - \alpha(t) | > 2\epsilon_k$ for some k sufficiently large. Let δ be the minimum between δ_k and a δ' corresponding to ϵ_k in the uniform continuity of $\alpha(t)$; choose r_i such that $| t - r_i | < \delta$; then $| \alpha(t) - \alpha(r_i) | < \epsilon_k$ and $| g(t) - \alpha(r_i) | = | g(t) - \alpha(t) + \alpha(t) - \alpha(r_i) | \geq | g(t) - \alpha(t) | - | \alpha(t) - \alpha(r_i) | > 2\epsilon_k - \epsilon_k = \epsilon_k$, in contradiction with (6).

LEMMA 5. *Let* $x \in B$ *and* $f(x., \cdot) \in L_p([0, a], \mathbf{R}^n)$. *Then for all* $t \in [0, a]$, $\int_0^t G[x, s]ds$ *exists and its class contains a continuous function. Moreover*

$$[\textstyle\int_0^t G(x, s)ds](\tau) = \int_0^\beta f(x_s, s)ds, \textit{ for } \tau \in [0, a] \textit{ where}$$

$$\beta = \min (\tau, t).$$

Proof. In order to simplify the details, we will prove the Lemma in the case $\mathbf{R}^n = \mathbf{R}$. The general case follows using projection operators. Now, by Lemma 4, it is sufficient to prove that for $t \in [0, a]$, $\epsilon > 0$ there exists a $\delta > 0$ such that for $\tau_0 \in [0, a]$

$$(7) \qquad | (\textstyle\int_0^t G(x, s)ds)(\tau) - \int_0^\beta f(x_s, s)ds)| < \epsilon$$

for almost every $\tau \in [\tau_0 - \delta, \tau_0 + \delta] \cap [0, a]$. Given $\epsilon > 0$ choose $\delta > 0$ such that $\int_{t_1}^{t_2} | f(x_s, s) | ds < \epsilon/2$ if $| t_1 - t_2 | < 2\delta$ and $0 \leq t_1 \leq t_2 \leq a$. Let $I = [\tau_0 - \delta, \tau_0 + \delta] \cap [0, a]$. We consider two cases, $t \leq \tau_0 - \delta$ and $t > \tau_0 - \delta$. In the first case we know by Lemma 3 that $[\int_0^t G(x, s)ds]_I = \int_0^t G(x, s)_I ds$ is a class that contains the constant function with value $\int_0^t f(x_s, s)ds$, and (7) follows. In the case $t > \tau_0 - \delta$, the relation (7) is reduced to

$$(8) \qquad | (\textstyle\int_{\tau_0-\delta}^t G(x, s)_I ds)(\tau) - \int_{\tau_0-\delta}^\beta f(x_s, s)ds | < \epsilon,$$

since by a similar argument as before the integrals between 0 and $\tau_0 - \delta$ cancel each other.

Now in order to prove (8) we have to consider two cases

a) $t \geq \tau_0 + \delta$

b) $t < \tau_0 + \delta$

In case a), (8) reduces to

$$| (\int_{\tau_0-\delta}^{\tau_0+\delta} G(x, s)_I ds)(\tau) - \int_{\tau_0-\delta}^{\tau} f(x_s, s)ds | < \epsilon. \tag{9}$$

To prove (9) it is enough to show that

$$| (\int_{\tau_0-\delta}^{\tau_0+\delta} G(x, s)_I ds)(\tau) | < \epsilon/2$$

almost everywhere in I. By (3) we have that

$$| f(x_s, s) | - G(x, s)_I(\tau) \geq 0$$

almost everywhere in I. Therefore, approximating the function $\varphi(s)(\tau) = | f(x_s, s) | - G(x, s)_I(\tau)$, $\tau \in I$, by means of simple functions whose values are nonnegative functions (of $\tau \in I$), we deduce

$$(\int_{\tau_0-\epsilon}^{\tau_0+\delta} G(x, s)_I ds)(\tau) \leq \int_{\tau_0-\delta}^{\tau_0+\delta} | f(x_s, s) | ds < \epsilon/2.$$

In a similar way one proves

$$(\int_{\tau_0-\delta}^{\tau_0+\delta} G(x, s)_I ds)(\tau) > -\epsilon/2,$$

proving case a). Case b) is proved similarly.

Proofs of the theorems

Proof of Theorem 1. By hypothesis

$$x(t) = x_0(0) + \int_0^t f(x_s, s)ds,$$

for $0 \leq t \leq a$, since $x_t(\tau) = x(t)$ for $0 \leq t \leq \tau \leq a$ and $x_t(\tau) = x(\tau)$ for $0 \leq \tau < t \leq a$, we have $x_t(\tau) = x_0(0) + \int_0^\beta f(x_s, s)ds$, where $\beta = \min(t, \tau)$, $0 \leq t \leq a$ and $0 \leq \tau \leq a$. By Lemma 5

$$x_t(\tau) = x_0(0) + [\int_0^t G(x, s)ds](\tau)$$

for $0 \leq \tau \leq a$ and $0 \leq t \leq a$. By (3) $G(x, s) = G(x_s, s)$ and therefore

$$y(t)(\tau) = y_0 + [\int_0^t G(y(s), s)ds](\tau)$$

proving the theorem.

Proof of Theorem 2. By hypothesis

$$y(t) = x_0 + \int_0^t G(y(s), s)ds \tag{12}$$

for $t \in [0, a]$ where $y(t) \in B$. From the properties of B there follows the existence of a function $x \in B$ such that $x(\tau) = x_0(\tau)$ for all $\tau \in [-h, 0)$ and $x(\tau) = y(\tau)(\tau)$ if $\tau \in [0, a]$. Moreover $x_t \in B$ for all $t \in [0, a]$.

We now prove that $x_t = y(t)$ for all $t \in [0, a]$. Fix $t \in [0, a]$, then if $\tau \in [0, t]$, (12) and Lemma 2 imply

$$[y(t)]_{[0,\tau]} = (x_0)_{[0,\tau]} + \int_0^t G[y(s), s]_{[0,\tau]} ds.$$

By (3)

$$\int_0^t G[y(s),\ s]_{[0,\tau]}ds = \int_0^\tau G[y(s),\ s]_{[0,\tau]}ds.$$

Therefore $y(t)(\tau) = y(\tau)(\tau) = x(\tau) = x_t(\tau)$ if $-h \leq \tau < t$.

When $\tau \in [t, a]$, (12) and Lemma 3 imply $y(t)(\tau) = y(t)(t) = x(t) = x_t(\tau)$ and the relation $x_t = y(t)$ is proved.

By (3) we can substitute x for $y(s)$ in (12) and by Lemma 5 we get $y(t)(t) = x(t) = x_0(0) + \int_0^t f(x_\cdot, s)ds$, proving the theorem.

ESCUELA SUPERIOR DE FÍSICA Y MATEMÁTICAS, IPN
CENTRO DE INVESTIGACION DEL IPN
UNIVERSITY OF SOUTHERN CALIFORNIA

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