# A NOTE ON MINIMALITY AND INTERPOLATION OF HARMONIZABLE HILBERT SEQUENCES*

By Habib Salehi

In this note we give sufficient conditions for a harmonizable Hilbert sequence to be non-minimal or interpolable. In [2] H. Cramér gave a sufficient condition for the determinism of a harmonizable Hilbert sequence. Later J. L. Abreu [1] showed that any harmonizable Hilbert sequence is a projection of a stationary Hilbert sequence taking values in a larger Hilbert space. As a corollary Abreu obtained the sufficiency condition of Cramér. Here we use Abreu's representation to prove our results, although a direct proof based on the original paper of H. Cramér [2] is possible.

$Z$ denotes the set of integers. $\mathfrak{S}$ stands for closed linear span. Let $\mathcal{H}$ be a Hilbert space with inner product $(,)$ and let $x_n \in \mathcal{H}$ for $n \in Z$. Define $H_\infty(x) = \mathfrak{S}\{x_n, n \in Z\}$, $H_n(x) = \mathfrak{S}\{x_k, k \in Z, k \neq n\}$. The Hilbert sequence $\{x_n\}$ is said to be non-minimal if $H_n(x) = H_\infty(x)$ for each $n \in Z$. The covariance of a Hilbert sequence $\{x_n\}$ is defined by $r(m, n) = (x_m, x_n)$ for $m, n \in Z$. The Hilbert sequence $\{x_n\}$ is said to be stationary if its covariance depends only on $m$-$n$. A Hilbert sequence $\{x_n\}$ is called harmonizable [1], [2] if for some complex valued Borel measure $f$ on $[0, 2\pi] \times [0, 2\pi]$

$$r(m, n) = \tfrac{1}{4}\pi^2 \int_0^{2\pi} \int_0^{2\pi} e^{i(mx-ny)} df(e^{ix}, e^{iy}).$$

$f$ is called the spectral measure of $\{x_n\}$. A Hilbert sequence $\{x_n\}$ is stationary if and only if it is harmonizable and its spectral measure is concentrated on the diagonal of $[0, 2\pi] \times [0, 2\pi]$.

Theorem 1. *Let $\{x_n\}$ be a harmonizable Hilbert sequence with the spectral measure $f$. Let $g(e^{i\theta}) = \frac{1}{2}\pi \int_0^\theta \int_0^{2\pi} d\,|f(e^{ix}, e^{iy})|$. If $\int_0^{2\pi} (1/g'(e^{i\theta}))d\theta = \infty$, then $\{x_n\}$ is non-minimal.*

For the proof of this theorem we need the following lemma whose proof is similar to the proof of a proposition in [1] and hence is omitted.

Lemma. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces and $A:\mathcal{H} \to \mathcal{K}$ be a bounded linear operator. Let $\{x_n\} \subseteq \mathcal{H}$ be a non-minimal Hilbert sequence. Then $\{Ax_n\} \subseteq \mathcal{K}$ is also a non-minimal Hilbert sequence.*

*Proof of Theorem* 1. By the representation theorem in [1]

$$x_n = Py_n, n \in Z,$$

where $\{y_n\}$ is a stationary sequence in a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, $P$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ and the spectral measure of $\{y_n\}$ is given by $g$. A well-known result for stationary sequences [3], [4] says $\int_0^{2\pi} (1/g'(e^{i\theta}))d\theta = \infty$ if and only if $\{y_n\}$ is non-minimal. We use the above lemma to complete the proof.

* Supported in part by NSF GP-28658.

The rest of this note is devoted to establishing a sufficient condition for interpolability of a harmonizable Hilbert sequence. Let $J$ be a finite subset of $Z$. Following [5] we say that the Hilbert sequence $\{x_n\}$ is interpolable if for each finite subset $J$, $\mathfrak{S}\{x_n, n \in Z/J\} = H_\infty(x)$. We remark that in the setting of our lemma $\{Ax_n\}$ is interpolable whenever $\{x_n\}$ is interpolable. Also there is a well-known characterization for interpolability of stationary Hilbert sequence in terms of its spectral density [5]. Because of these facts, arguments similar to the ones in the proof of Theorem 1 can be given to establish the following theorem.

THEOREM 2. *Let* $\{x_n\}$ *be a harmonizable Hilbert sequence with the spectral measure* $f$. *Let* $g(e^{i\theta}) = \frac{1}{2}\pi \int_0^\theta \int_0^{2\pi} d\,|f(e^{ix}, e^{iy})|$. *If* $\int_0^{2\pi} (\,|p(e^{i\theta})|^2/g(e^{i\theta}))d\theta = \infty$ *for each non-zero trigonometric polynomial* $p$, *then* $\{x_n\}$ *is interpolable.*

MICHIGAN STATE UNIVERSITY

REFERENCES

[1] J. L. ABREU, *A note on harmonizable and stationary sequences*, Bol. Soc. Mat. Mexicana, **15**(1970), 48–51.
[2] H. CRAMÉR, On the structure of non-stationary stochastic processes, Proc. IV Berkeley Sym. Math. Stat. Prob., **2**, 57–78, Univ. Calif. Press, 1961.
[3] A. N. KOLMOGOROV, *Stationary sequences in Hilbert space*, Bull. Math. Univ. Moscow, **2**(1941) 1–40.
[4] P. R. MASANI, *The prediction theory of multivariate stochastic processes* III, Acta Math. **104**(1960), 142–62.
[5] H. SALEHI, *Application of the Hellinger integrals to q-variate stationary stochastic processes*, Ark. Mat. **7**(1967), 305–11.