COLIMITS AND COREFLECTIVE SUBCATEGORIES IN PARTIALLY ORDERED TOPOLOGICAL SPACES

BY O. C. GARCIA*

Introduction and Notation

In this paper we consider the categories **PTop** of all Partially Ordered Topological Spaces with continuous, isotone functions as their morphisms, and subcategories thereof.

We use the same notation as in [2], namely **Top** is the category of topological spaces, **H** of Hausdorff spaces, **KPTop** the full subcategory of **PTop** consisting of all the objects whose underlying topological space belongs to **K**. We denote further by **POTS** the full subcategory of **PTop** having a semi-continuous partial order, by **HOTS** the (full) subcategory of continuously partially ordered spaces [10], by **KPOTS** the intersection of **KPTop** and **POTS**, and by **KOTS** the intersection of **KPTop** and **HOTS**.

Section 1, deals with the colimits in these categories. Although final structures do not seem to exist for arbitrary families of maps, we find and characterize coproducts and coequalizers concluding that **PTop, HPTop, HPOTS** and **HOTS** are cocomplete.

Section 2, concerns itself with coreflective subcategories. In order to use 13.1.2 of [3], we recall from [2], that **PTop** is locally and colocally small, and we show that so is **HPTop.** The subcategories **HPOTS** and **HOTS** are also locally small. Moreover every nonempty object of **PTop** is a generator.

This leads to conclude that the subcategories **KOTS** which are coreflective in **HOTS** are exactly those for which **K** is coreflective in **H**. Similarly for **KPTop** as subcategories of **HPTop** and for **KPOTS** as subcategories of **HPOTS.**

Finally section 3 introduces left adjoints of the Inclusion and Forgetful functors to relate **Top** and **PTop**, giving thus more insight into reflective and coreflective subcategories of **PTop.** The Inclusion functor is left-adjoint to the Forgetful functor. If **K** is a subcategory of **H**, **K** is coreflective in **KOTS**. Therefore if **K** is coreflective in **H**, **K** is coreflective in **HOTS**. A similar result is obtained for some reflective subcategories **K** of **H**. A left adjoint of the Inclusion functor is obtained for epireflective subcategories of **H**. Finally we find an adjoint situation of functors **KOTS** $\rightleftarrows$ **HOTS** for **K** coreflective subcategory of **H**, and a similar result for the reflective case.

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1. Colimits

Let $(X_i, \leqslant_i)_{i \in I}$ be a family of objects in **PTop**. On the underlying set of the topological space $\amalg_{i \in I} X_i$ we define the following partial order: $(x, i) \leqslant (y, j)$ if and only if $i = j$ and $x \leqslant_i y$. Let $(s_i)_{i \in I}$ be the family of natural injections $s_j: X_j \to \amalg_{i \in I} X_i$ such that $s_j(x) := (x_i, j)$. As we know from [2] $(\amalg_{i \in I} X_i, \leqslant)$ together with $(s_i)_{i \in I}$ is the coproduct $\amalg_{i \in I}(X_i, \leqslant_i)$ in **PTop** and if each $\leqslant_i$ had been continuous or semicontinuous so would have been the partial order $\leqslant$. Accordingly **POTS** and **HOTS** have coproducts.

Remark: As is well known, every equivalence relation π in a topological space X determines a quotient space in **Top**.

The corresponding situation **Ptop** is not as simple.

LEMMA 1: *Let $(X, T_X, \leqslant) \in PTop$, and R be an equivalence relation on X. If a partial order $\leqslant_R$ on X/R is well defined by $a_R \leqslant_R b_R$ if and only if $a_R = b_R$ or $a \leqslant b$, then the natural map ν_R is a morphism and the quotient topological space with this partial order has the final structure with respect to ν_R.*

Proof: By the hypothesis on R and $\leqslant_R$, $a \leqslant b$ implies $a_R \leqslant_R b_R$ and consequently, ν_R in addition to being continuous is isotone. Let $Z \in$ **PTop** be arbitrary. If $g: X/R \to Z$ is a **PTop** morphism $g \circ \nu_R$ is continuous and isotone. Since X/R has the final topological structure with respect to ν_R, g is continuous. By using the definition of $\leqslant_R$ one easily sees that g is also isotone.

Remark: Let $f_i: X \to Y_i$ be a family of **PTop**-morphisms indexed by I. Let R be the intersection of all $\ker f_i$, $i \in I$. Define $\leqslant_R$ in X/R as follows: $a_R \leqslant_R b_R$ if and only if $f_i(a) \leqslant f_i(b)$ for all $i \in I$. Then $\leqslant_R$ is a well defined partial order on X.

We recall from [2] that **PTop** is colocally small and accordingly for X the class of epimorphisms $f: X \to Y$ can be represented by a set.

Definition 1: Let $X \in$ **PTop**. For every equivalence relation π on the underlying set of X, let $S(\pi)$ be a representative set of those epimorphisms $f: X \to Y$ such that $\pi \subset \ker f$, and let $R(\pi)$ denote the intersection of all the kernels of maps in $S(\pi)$. We call an epimorphism $f: X \to Y$ in **PTop** a *special quotient* if there exist an equivalence relation π on X such that $f: X \to Y$ is equivalent to $\nu: X \to X/R(\pi)$ as an epimorphism, where ν is the natural map and the order on $X/R(\pi)$ is the one induced by the set of **PTop**-epimorphisms, $S(\pi)$.

LEMMA 2: *The category* **PTop** *has coequalizers.*

Proof: Let $f, g: X \to Y$ be given in **PTop**.

Let $\pi = \{(f(x), g(x)) \mid x \in X\} \cup \{(g(x), f(x)) \mid x \in X\} \cup \Delta_Y$. The set $S(\pi)$ of the above definition is nonempty as one map of the type $Y \to \{p\}$ belongs to it. We denote by $(\psi_i)_{i \in I}$ the family of **PTop**-epimorphisms in $S(\pi)$. Let $R = \bigcap_{i \in I} \ker \psi_i$ and let $(Y/R, \leqslant R)$ be the special quotient induced by the family $(\psi_i)_{i \in I}$. We now show that $\nu: Y \to Y/R$ is the coequalizer of f and g. Let

$x \in X$. By definition of π, $f(x)\pi g(x)$. Since $\pi \subset R$, $f(x)Rg(x)$ which we can write as $\nu \circ f(x) = \nu \circ g(x)$. Therefore $\nu \circ f = \nu \circ g$.

Suppose $h:Y \to Z$ has been given such that $h \circ f = h \circ g$. Then $\pi \subset \ker h$ and we obtain $R \subset \ker h$. Without loss of generality, let h be an epimorphism of $(\psi_i)_{i \in I}$. Define $k:Y/R \to Z$ by $k(\nu(y)) = h(y)$. It is easy to see that k is a well defined continuous and isotone map which makes the following diagram commutative:

$$\begin{array}{ccccc} X & \underset{g}{\overset{f}{\longrightarrow}} & Y & \overset{\nu}{\longrightarrow} & Y/R \\ & & {\scriptstyle h}\downarrow & \swarrow{\scriptstyle k} & \\ & & Z & & \end{array}$$

Since ν is surjective, k is unique.

Remark: It should be noted that the underlying topological space of Y/R may not be the coequalizer of the underlying topological spaces of X and Y, with the maps f, g.

THEOREM 1: *The coequalizers in* **PTop** *are exactly the special quotients.*

Proof: From the proof of Lemma 2 we have seen that every coequalizer is an special quotient. Without loss of generality let $\nu:X \to X/_K$ be an special quotient where $K = \ker \nu$, ν is the natural map, and the order on X/K is the one induced by the family of all types of **PTop**-epimorphisms m with domain X and $K \subset \ker m$. Let $i:K \to X\amalg X$ be the inclusion map. It is obvious now that ν is the coequalizer of $p_1 \circ i$ and $p_2 \circ i$ where p_1, p_2 are respectively the first and second projections.

THEOREM 2: *An epimorphism $f:X \to Y$ is an special quotient in* **PTop** *if and only if for every* **PTop**-*morphism $g:X \to Z$ such that* $\ker f \subset \ker g$, *there exists a unique* **PTop**-*morphism $h:Y \to Z$ which makes the following diagram commutative*:

$$\begin{array}{ccc} X & \overset{f}{\longrightarrow} & Y \\ {\scriptstyle g}\downarrow & \swarrow{\scriptstyle h} & \\ Z & & \end{array}$$

Proof: The necessity is obvious. Let S be a representative family of all **PTop**-epimorphisms m with domain X and $\ker f \subset \ker m$ and let $\nu:X \to X/R$ be the special quotient induced by S. By hypothesis there is a unique **PTop**-morphism $h:Y \to X/R$ such that $h \circ f = \nu$. We apply the first part of this proposition to the special quotient $\nu:X \to X/R$ and f and obtain a unique map $k:X/R \to Y$ such that $k \circ \nu = f$. Since both f and ν are surjective and therefore epimorphisms, from $h \circ k \circ \nu = \nu$ and $k \circ h \circ f = f$, $h \circ k = 1_{X/R}$ and $k \circ h = 1_Y$ follow. We conclude that f is equivalent to ν and is therefore a special quotient.

Remark: If $f{:}X \to Y$ is a surjective **PTop**-morphism where $f(a) \not\leqslant f(b)$ implies $a \not\leqslant b$, and if we define $\leqslant$ in $X/\ker f$ by $\nu_f(a) \leqslant \nu_f(b)$ if and only if $f(a) \leqslant f(b)$, then $\nu_f{:}X \to X/\ker f$ is a coequalizer. Indeed we know from [2] that there is an equalizer h of $\nu_f \circ p_1$ and $\nu_f \circ p_2$.

$$K \xrightarrow{h} X \Pi X \underset{p_2}{\xrightarrow{p_1}} X \xrightarrow{\nu_f} X/\ker f$$

One checks that ν_f is now coequalizer of $p_1 \circ h$ and $p_2 \circ h$.

In order to use 13.1.2 of [3] in Section 2, we conclude now:

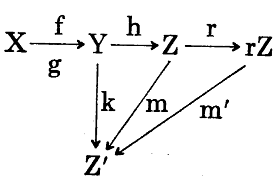
COROLLARY 1: *The category* **PTop** *is cocomplete.*

Proof: We have recalled from [2] that **PTop** has coproducts and we have shown in Lemma 2 that it has coequalizers. The result follows from a well known theorem of category theory. See for example [5], [7].

THEOREM 3: *The categories* **HPTop**, **HPOTS** *and* **HOTS** *are complete, cocomplete and locally small.*

Proof: From the proof of [2]-Theorem 1, one sees that these categories are complete. The same argument of [2]-Lemma 4, yields that the monomorphisms are injective and proves that these categories are locally small. We have already remarked that they are closed with respect to coproducts. All we need to show then, is that they are closed with respect to coequalizers. For convenience let us denote any one of the categories **HPTop, HPOTS** and **HOTS** by **SC**.

Let $f, g{:}X \to Y$ be two different **SC**-morphisms. Let $h{:}Y \to Z$ be their **PTop**-coequalizer. Since Z may not be Hausdorff or may not have semicontinuous (continuous) partial order, using [2]-Theorem 1, we take the **SC**-reflection rZ of Z. Let $r{:}Z \to rZ$ be the reflection map. The map $r \circ h$ is the coequalizer of f, g in **SC**. Indeed, consider the following diagram:



Since h is **PTop**-coequalizer of f, g, $h \circ f = h \circ g$. Therefore $(r \circ h) \circ f = (r \circ h) \circ g$. Suppose $k{:}Y \to Z'$ has been given such that $k \circ f = k \circ g$. Since $k \in$ **PTop** and h is a **PTop**-coequalizer, there exists a unique continuous isotone map m such that $m \circ h = k$. Since r is the **SC**-reflection, there exists a unique $m{:}rZ \to Z'$ in **SC**, such that $m' \circ r = m$. Therefore $m' \circ (r \circ h) = k$. The uniqueness of m' is clear.

2. Coreflective subcategories

THEOREM 4: *The category* **HPTop** *is colocally small.*

Proof: We first show that if $Y \in$ **HPTop**, every proper closed subspace U of Y is an equalizer. Given one such $U \subset Y \in$ **HPTop**, define: $R = \{((u, 1), (u, 2)) \mid u \in U\} \cup \{((u, 2), (u, 1)) \mid u \in U\} \cup \Delta_{Y \amalg Y}$. R is clearly an equivalence relation in $Y \amalg Y$ and the relation $\leqslant_R$ defined by $(x, i)_R \leqslant_R (y, j)_R$ *if and only if* $(x, i) \leqslant (y, j)$ *or there exists* $u \in U$ *such that* $x \leqslant u$ *and* $u \leqslant y$ is a partial order in $(Y \amalg Y)/R$. We call $Z: = ((Y \amalg Y)/R, \leqslant_R)$ and show that it is Hausdorff. Let $(x, i)_R$, $(y, j)_R$ be two distinct points of Z. If $x = y$, then $i \neq j$ and x, y belong to the complement of U, CU. We obtain then two disjoint saturated neighborhoods $CU \times \{i\}$ and $CU \times \{j\}$ of (x, i) and (y, j) respectively.

If $x \neq y$, there exist V, W disjoint open neighborhoods of x and y respectively, since Y is Hausdorff, and we obtain with them the disjoint saturated open sets $V \times \{i, j\}$ and $W \times \{i, j\}$ which are neighborhoods of (x, i) and (y, j) respectively. From [1] Chapter 1 it follows that Z is Hausdorff.

Consider $U \xrightarrow{i} Y \underset{\sigma_2}{\xrightarrow{\sigma_1}} Y \amalg Y \xrightarrow{\nu_R} Z$, and call $f = \nu_R \circ \sigma_1$ and $g = \nu_R \circ \sigma_2$.

It is clear that i is equalizer of f and g.

Having shown that every proper closed subspace of an space in **HPTop** is an equalizer, the closure of the image of an epimorphism $e: X \to Y$ can not be a proper subset of Y. Therefore $\overline{\overline{Y}} \leqslant 2^{2^{\overline{\overline{X}}}}$. See [8]

Remark: For the proof of out next lemma we shall use the well-known fact that for a category **C** closed with respect to coproducts, an object G is a generator if and only if for each object A in **C** there is an epimorphism $e_G: \amalg_{i \in I} G \to A$, where $\amalg_{i \in I} G$ denotes the coproduct of as many copies of G as I has elements.

LEMMA 3: *Every nonempty object of* **PTop** *is a generator.*

Proof: Let G be a nonempty object of **PTop** and $A \in$ **PTop** arbitrary. Let I be the set of **PTop**-morphisms $G \to A$. We define $e_G: \amalg_{i \in I} G \to A$ by $e_G(v, x) = v(x)$ for every $v \in I$, $x \in G$. This map is clearly continuous and isotone. Let $a \in A$. If $\bar{a}: G \to A$ is the constant map with value a, $\bar{a} \in I$ and $e_G(\bar{a}\, x) = \bar{a}(x) = a$ for any $x \in G$ which is nonempty. It follows that e_G is surjective and accordingly an epimorphism.

Remark: Since **HPTop**, **HPOTS** and **HOTS** are cocomplete and therefore have coproducts, the argument of Lemma 3 applies and shows that every nonempty object in these categories is a generator.

LEMMA 4: *If* **K** *is a subcategory of* **Top**, **KPTop** *is closed with respect to limits in* **PTop** *if and only if* **K** *is closed with respect to limits in* **Top**. *Morover* **KPTop** *is closed with respect to colimits in* **PTop** *if and only if* **K** *is closed with respect to colimits in* **Top**.

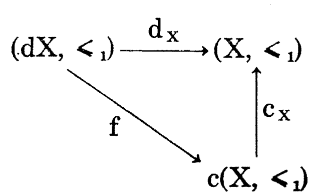
Proof: The proof of the first statement follows directly from the construction of products and equalizers in [2], and the observation that **K** is a subcategory of **KPTop** and **Top** of **PTop**. The converse of the second statement is also easy using the argument of Theorem 1, but one needs to be aware that the forgetful functor do not preserve coequalizers, as we have remarked before.

THEOREM 5: *If K is a subcategory of* **H** (**Top**), *the following statements are equivalent*:

1) **K** *is coreflective in* **H**
2) **KPTop** *is coreflective in* **HPTop**
3) **KPOTS** *is coreflective in* **HPOTS**
4) **KOTS** *is coreflective in* **HOTS**.

Proof: By a direct application of 13.1.2 of [3], theorem 3, theorem 4, lemma 3 and lemma 4 yield that 1) and 2) are equivalent. Suppose **KPTop** is coreflective in **HPTop**. Let $(X, \leqslant) \in$ **HPOTS** and $c_X{:}c(X, \leqslant_1) \to (X, \leqslant_1)$ be its **KPTop**-coreflection. We shall show that $c(X, \leqslant_1)$ has a semicontinuous partial order. Let $d_X{:}dX \to X$ be the **K**-coreflection of X.

Since the coreflections in **H** are bijective, we can assume without loss of generality that the underlying sets of X, dX, $c(X, \leqslant_1)$ are all the same, and that the graph of the maps c_X and d_X is the diagonal. Since $(dX, \leqslant_1) \in$ **KPTop**, and the map $d_X{:}(dX, \leqslant_1) \to (X, \leqslant_1)$ is in **HPTop**, there exists a unique continuous and isotone map f which makes the following diagram commutative:



Let $a \nleqslant b$ in $c(X, \leqslant_1)$. If $c_X(a) \leqslant_1 c_X(b)$, then $d_X^{-1}c_X(a) \leqslant d_X^{-1}c_X(b)$ and $a = fd_X^{-1}c_X(a) \leqslant fd_X^{-1}c_X(b) = b$ which is a contradiction. Therefore $c_X(a) \nleqslant c_X(b)$. Since $\leqslant_1$ is semicontinuous in $(X, \leqslant_1)$, we find two open neighborhoods, U of $c_X(a)$ and V of $c_X(b)$ such that $c_X(a) \nleqslant v$ and $u \nleqslant c_X(b)$ for all $u \in U$ and $v \in V$. This shows that $U = c_X^{-1}U$ and $V = c_X^{-1}V$ are two open neighborhoods in $c(X, \nleqslant_1)$ such that $a \nleqslant v$ and $u \nleqslant b$ for all $u \in U$ and $v \in V$. Accordingly the partial order of $c(X, \leqslant_1)$ is semicontinuous.

We have shown that if $(X, \leqslant_1) \in$ **HPOTS**, given an arbitrary $f{:}Y \to (X, \leqslant_1)$ in **HPTop** such that $Y \in$ **KPTop**, there exists $c(X, \leqslant_1) \in$ **KPOTS**, $c_X{:}c(X, \leqslant_1) \to (X, \leqslant_1)$ in **HPOTS** and $\bar{f}$ continuous and isotone such that $c_X \circ \bar{f} = f$. This will be true in particular whenever $Y \in$ **KPOTS** $\subset$ **KPTop**.

The same argument shows that 2) implies 4).

Let **KPOTS** be coreflective in **HPOTS**. If $X \in$ **HPOTS** has the discrete order, it is clear that its **KPOTS**-coreflection has the discrete order. (The coreflection map is bijective.) Therefore **K** is coreflective in **H**. Similarly 4) implies 1).

Corollary 2: *Let K be a subcategory to Top. The following statements are equivalent*:

1) **K** *is coreflective in* **Top**
2) **KPTop** *is coreflective in* **PTop**
3) **KPOTS** *is coreflective in* **POTS**
4) **KOTS** *is coreflective in* **HOTS.**

Proof: The proof is verbatim as in Theorem 5. In the proof of our next theorem we shall use the following obvious lemma:

Lemma 5: *Let* **S** *be a subcategory of* **B** *and* **A** *a subcategory of* **S**. *If* **A** *is coreflective in* **B, A** *is coreflective in* **S.**

Theorem 6: *Let* **K** *be a subcategory of* **Top(H)**. *Then* **K** *is coreflective in* **PTop** **(HPTop)** *if and only if* **K** *is coreflective in* **Top(H)**.

Proof: The necessity follows from Lemma 5. Conversely if we consider **K** as a subcategory of **PTop** the coproducts and coequalizers of **K** in **PTop** will have the discrete order and therefore belong to **K.** Similarly for **HPTop.**

3. Leftadjoints of the inclusion and forgetful functors

Let **K** be a subcategory of **H**, and $X \in$ **K**, by associating to X the **PTop** space (X, d) where d is the discrete partial order (no two elements are comparable), we introduce the inclusion functor U:**K** $\rightarrow$ **KOTS**.

Theorem 6: *Let* **K** *be a subcategory of* **H**, F:**KOTS** $\rightarrow$ **K** *the order-forgetful functor,* U:**K** $\rightarrow$ **KOTS** *the inclusion functor. Then U is left-adjoint of F.*

Proof: Let $f:A \rightarrow B$ in **K** and $g:C \rightarrow D$ in **KOTS** be given. Define $\eta_{B.C}$:**KOTS**-$(UB, C) \rightarrow K(B, FC)$ by $\eta_{B,C}(h)(b) = h(b)$. Since $\eta_{B,C}$ is clearly a bijection, we only need to show that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{KOTS}(UB, C) & \xrightarrow{\eta_{B,C}} & \mathbf{K}(B, FC) \\ {\scriptstyle \mathbf{KOTS}(Uf, g)}\downarrow & & \downarrow{\scriptstyle \mathbf{K}(f, Fg)} \\ \mathbf{KOTS}(UA, D) & \xrightarrow[\eta_{A,D}]{} & \mathbf{K}(A, FD) \end{array}$$

Let $h \in \mathbf{KOTS}(UB, C)$ and $a \in A$ be arbitrary.
Then $(\mathbf{K}(f, Fg) \circ \eta_{B,C})\,(h)(a) = \mathbf{K}(f, Fg)\,(\eta_{B,C}(h))\,(a)$

$$= (Fg \circ \eta_{B,C}(h) \circ f)(a) = Fg(\eta_{B,C}(h)(f(a)))$$

$$= Fg(h(f(a))) = g(h(f(a))) = (g \circ h \circ Uf)(a)$$

$$= \eta_{A,D}(g \circ h \circ Uf)(a) = (\eta_{A,D} \circ \mathbf{KOTS}(Uf, g))(h)(a)$$

Since a and h are arbitrary the diagram commutes and η is a natural equivalence as required.

COROLLARY 3: *Every subcategory* **K** *of* **H** *is coreflective in* **KOTS.**

Proof: Follows directly from [3], 8.1

COROLLARY 4: *If K is coreflective in H, K is coreflective in* **HOTS.**

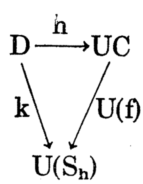
Proof: Let $X \in$ **HOTS**, let (X_1, c_1) be the **KOTS**-coreflection of X and (X_2, c_2) the **K**-coreflection of (X_1, c_1). It is an easy matter to check that (X_2, c_2) is the **K**-coreflection of X.

It is easy to find an example to show that the Forgetful functor F is not a left-adjoint of U. However we have the following

LEMMA 6: *The inclusion functor U:***K** $\rightarrow$ **KOTS** *has a left-adjoint for the following subcategories* **K** *of* **H**:

1) *Hausdorff*
2) *completely regular*
3) *compact*
4) *real compact*
5) *zero dimensional*
6) *boolean spaces.*

Proof: Since all these subcategories of **H** are productive and closed hereditary, they have equalizers and products and are therefore complete. As an embedding, U preserve limits. By [7] Theorem 2 page 110, we need only show that for every $D \in$ **KOTS**, there exists a set S_D of **K**-objects, which is a solution set of D with respect to U. Choose a set D' such that $\bar{\bar{D}}' = 2^{2^{\bar{\bar{D}}}}$. Define $S_D = \{(S, t) \mid S \subset D'$ and t is a topology on $S\}$. We shall show that S_D is a solution set. Let $D \xrightarrow{h} UC$ be given. Then $\overline{\overline{\Gamma Imh}} \leqslant D'$. (We mean by ΓA the closure of the set A.) Choose a subset S_h of D' such that $\bar{\bar{S}}_h = \overline{\overline{\Gamma Imh}}$, find a bijection $b{:}S_h \rightarrow \Gamma Imh$ and induce on S_h the topological structure from ΓImh. We obtain in this way that $S_h \in S_D$ and b is an homeomorphism. Call h' the map $D \rightarrow U(\Gamma Imh)$ defined by $h'(d) = h(d)$, i the inclusion $\Gamma Imh \rightarrow C$, and define k, f so that $k = U(b^{-1}) \circ h'$ and $f = i \circ b$. Now $Uf \circ k = U(i \circ b) \circ k = U(i \circ b) \circ U(b^{-1}) \circ h = U(i) \circ h' = h$, $S_h \in S_D$ and the following diagram commutes:



Therefore S_D is a solution set as required.

COROLLARY 5: *For the subcategories* **K** *of* **H** *listed in Lemma* 6, **K** *is reflective in* **KOTS.**

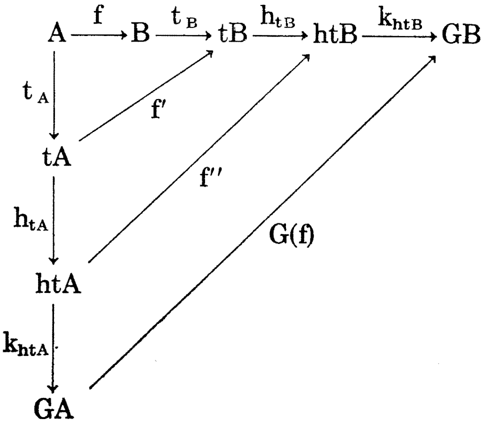
Proof: See [3], 8.1.

Let $Y \in$ **KOTS**. Define in Y, $a \sim b$ if and only if $\{a, b\}$ has an upper bound or a lower bound, and $a \pi_Y b$ if and only if there exists $a_1, \cdots, a_n \in Y$ such that $a \sim a_1, \cdots, a_n \sim b$. Then π_Y is an equivalence relation on Y.

If **K** is an epireflective subcategory of **H** and $Y \in$ **KOTS**, let tY denote the topological space Y/π_Y and h:**Top** $\rightarrow$ **H** and k:**H** $\rightarrow$ **K** the epireflectors. We see that the natural map $Y \rightarrow Y/\pi_Y$ is isotone as tY has been obtained by identifying all the points which can be compared or are extrema of chains $c_1, \cdots, c_n$ where c_i can be compared to c_{i+1}. Let GY denote $khtY$. We define so a functor G:**KOTS** $\rightarrow$ **K** as will be shown in the next Theorem.

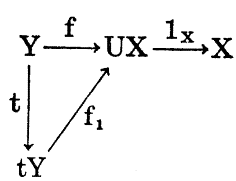
THEOREM 7: *Let* **K** *be an epireflective subcategory of* **H**. *Then G can be constructed into the left-adjoint of U.*

Proof: G is certainly a functor. Given $A \xrightarrow{f} B$ in **KOTS**, we consider:



For $t_B \circ f$ and t_A, f' is unique. For $h_{tB} \circ f'$ and h_{tA}, f'' is unique and for $k_{htB} \circ f''$ and k_{htA}, $G(f)$ is well defined. It is an easy routine to check all the functor properties of G. For every $Y \in$ **KOTS**, call $g_Y = k_{htY} \circ h_{tY} \circ t_Y$. Let $X \in$ **K** and $Y \in$ **KOTS** and define $\lambda_{Y,X}$ as follows: $\lambda_{Y,X}$:**K**$(GY, X) \rightarrow$ **KOTS**(Y, UX) and $\lambda_{Y,X}(f) = f \circ g_Y$. Since t_Y is a natural map and h and k are epireflections, g_Y is continuous and isotone. Therefore $f \circ g_Y \in$ **KOTS**(Y, UX). We claim that λ is a natural equivalence.

Let $\lambda_{Y,X}(a) = \lambda_{Y,X}(b)$. Then $a \circ g_Y = b \circ g_Y$. Since t_Y is surjective and h, k epireflections, we obtain that $a = b$. Therefore $\lambda_{Y,X}$ is injective. Let f:$Y \rightarrow UX$. To show that $\pi_Y \subset \ker f$, let $a, b \in Y$ and $a \pi_Y b$. If $a \sim b$ since UX has discrete order, $f(a) = f(b)$. If $a \sim a_1$, $a_1 \sim a_2, \cdots, a_n \sim b$, similarly $f(a) = f(b)$. Therefore there exists f_1 such that the following diagram commutes:



Moreover, since h and k are epireflectors, there exist f_2 and f_3 such that $f_2 \circ h = f_1$ and $f_3 \circ k = f_2$. Therefore $f_3 \circ k \circ h \circ t = f_2 \circ h \circ t = f_1 \circ t = f$, which can be rewritten as $f = f_3 \circ g_Y = \lambda_{Y,X}(f_3)$, and shows that $\lambda_{Y,X}$ is surjective.

To show that λ is natural, let $A \xrightarrow{c} B$ in **K** and $C \xrightarrow{i} D$ in **KOTS**. Consider the following diagrams:

$$
\begin{array}{ccc}
\mathbf{K}(GD, A) & \xrightarrow{\lambda_{D,A}} & \mathbf{KOTS}(D, UA) \\
{\scriptstyle \mathbf{K}(Gi,\, c)}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{KOTS}\,(i,\, Uc)} \\
\mathbf{K}(GC, B) & \xrightarrow[\lambda_{C,B}]{} & \mathbf{KOTS}(C, UB)
\end{array}
\qquad
\begin{array}{ccc}
a & \longmapsto & a \circ g_D \\
\big\downarrow & & \big\downarrow \\
 & & Uc \circ a \circ g_D \circ i \\
c \circ a \circ Gi & \longmapsto & c \circ a \circ Gi \circ g_C
\end{array}
$$

By the definition of Gi, we know that $Gi \circ g_C = g_D \circ i$.
Since Uc is the same map as c, we obtain $Uc \circ a \circ g_D \circ i = c \circ a \circ Gi \circ g_C$.

COROLLARY 6: *Let* **K** *be a coreflective subcategory of* **H**, C:**H** $\to$ **K** *the correflector and* E:**K** $\to$ **H** *the inclusion functor. Then UEG is left-adjoint to UCF.*

Proof: Consider the adjoint situations

$$
\mathbf{KOTS} \underset{U}{\overset{G}{\rightleftarrows}} \mathbf{K} \underset{C}{\overset{E}{\rightleftarrows}} \mathbf{H} \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{HOTS}.
$$

A similar result holds for the reflective subcategories **K** of **H**.

CENTRO DE INVESTIGACIÓN DEL IPN

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