ASSOCIATED PRIME DIVISORS IN THE SENSE OF NOETHER

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1. Introduction

Recently, several papers appeared discussing various notions of associated prime divisors (see [1], [2], and [5]). In this note we produce a sixth type, which, in the case of Noetherian rings, reduces to the associated primes (the prime radicals of the primary ideals in a normal primary decomposition). In [3], we discussed a concept called relatively prime ideals in the sense of Noether (B is called relatively prime to A in the sense of Noether, if $A:B = A$). It turns out that in Noetherian rings, P is an associated prime of A, if and only if, P is not relatively prime to the P-isolated component of A in the sense of Noether.

Throughout this note, R denotes a commutative ring with unity, and all ideals and elements are assumed to be in such a ring. On the whole, our terminology will be that of [6]. We use the concept ideal in the somewhat restrictive sense, in that for us, an ideal is not the entire ring. We shall let upper case letters, most frequently the beginning of the alphabet, denote ideals, and lower case letters, elements of R. For us, $A:x$ will denote the ideal quotient $A:(x)$, and $A(P)$ will denote the P-isolated component of the ideal A, i.e., $A(P) = \{x \in R \mid$ there exists $y \notin P$ such that $xy \in A\}$. If A is an ideal, we let $Z(A)$ denote the set of all zero divisors modulo A, i.e., $Z(A) = \{x \in R \mid$ there exists $y \notin A$ such that $xy \in A\}$.

2. Definitions and Preliminary Results

Definition 1: If A and B are ideals, then B is called relatively prime to A in the sense of Noether if $A:B = A$.

2: Let P be a prime ideal. Then P is called an associated prime divisor of A in the sense of Noether, if P is not relatively prime to $A(P)$ in the sense of Noether. We denote this condition by (Ne) and say P is a (Ne)-prime of A, if P satisfies (Ne) relative to A.

We first show that there are several ways of characterizing the (Ne)-primes of A.

PROPOSITION 1: *Let P be a prime ideal containing A. Then the following statements are equivalent:*

(a) *P is a (Ne)-prime of A.*
(b) $A(P):P \neq A(P)$.
(c) $A(P):P > A(P)$.
(d) *There exists $x \notin A(P)$ such that $A(P):x = P$.*

Proof. (a) $\Leftrightarrow$ (b) This equivalence is clear from the definitions.
(b) $\Leftrightarrow$ (c) This equivalence is clear since $A(P) \subset A(P):P$.

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(c) $\Rightarrow$ (d) Let $x \in A(P):P\backslash A(P)$. Then $xP \subset A(P)$, so $A(P):x \supset P$. Now let $t \in A(P):x$. Then $tx \in A(P)$, and since $x \notin A(P)$, this means $t \in P$. Hence $A(P):x = P$.

(d) $\Rightarrow$ (c) Suppose there exists $x \notin A(P)$ such that $A(P):x = P$. Then $xP \subset A(P)$, hence $x \in A(P):P$. But $x \notin A(P)$ implies that $A(P):P > A(P)$.

We now record a special property of the ideal quotient $A(P):P$.

PROPOSITION 2. *Let P be a prime ideal containing A. Then either $A(P):P \subset P$, or $A(P) = P$ in which case $A(P):P = R$.*

Proof. Assume $A(P):P \not\subset P$. Then there exists $t \notin P$ such that $tP \subset A(P)$. Thus for each $p \in P$, $tp \in A(P)$, and since $t \notin P$, it follows that $p \in A(P)$. Hence $P \subset A(P)$, and since the other containment relation is always true, it follows that $A(P) = P$.

Minimal prime overideals play an important part in many discussions in commutative ideal theory. We now produce a result characterizing these minimal prime overideals.

PROPOSITION 3. *Let P be a prime ideal containing A. Then the following statements are equivalent:*

(a) *P is a minimal prime overideal of A.*

(b) *$A(P)$ is a P-primary ideal.*

(c) *$\sqrt{A(P)} = P$.*

(d) *There exists $x \notin P$ such that $A(P):x$ is a P-primary ideal.*

Proof. (a) $\Leftrightarrow$ (b) This is proposition 6 of [2].

(b) $\Rightarrow$ (c) This implication is clear from the definition of primary ideals.

(c) $\Rightarrow$ (b) If $\sqrt{A(P)} = P$, then to show $A(P)$ is P-primary, we need only show that $xy \in A(P)$ and $x \notin P$ implies $y \in A(P)$. But this follows from the definition of $A(P)$.

(b) $\Rightarrow$ (d) Let $x = 1$.

(d) $\Rightarrow$ (c) Let $p \in P$. Then $p^n \in A(P):x$ for some positive integer n. Thus $p^n d \in A(P)$, and since $x \notin P$, we have that $p^n \in A(P)$. Hence $\sqrt{A(P)} = P$.

We end this section with a proposition concerning the P-component of an ideal and the operations of $\cap$, :, and $\sqrt{\ }$.

PROPOSITION 4. *Let A and B be ideals, and P a prime ideal. Then:*

(a) *$(A \cap B)(P) = A(P) \cap B(P)$. Consequently, the P-component distributes over any finite intersection.*

(b) *If B is finitely generated, then $(A:B)(P) = A(P):B$.*

(c) *$\sqrt{A(P)} = (\sqrt{A})(P)$.*

Proof. (a) This is proposition 7 of [2].

(b) We first show that $(A:x)(P) = A(P):x$. To see this we observe the following: $y \in (A:x)(P)$ if and only if there exists $t \notin P$ such that $yt \in A:x$ if and only if $ytx \in A$ for some $t \notin P$ if and only if $yx \in A(P)$ if and only if

$y \in A(P):x$. Now suppose $B = (b_1, \cdots, b_n)$. Then $(A:B)(P) = (A:(b_1, \cdots, b_n))(P) = (\bigcap_1^n A:b_i)(P) = \bigcap_1^n((A:b_i)(P)) = \bigcap_1^n(A(P):b_i) = A(P):B$.

(c) Let $x \in \sqrt{A(P)}$. Then there exists positive integer n such that $x^n \in A(P)$. So $x^n t \in A$ for some $t \notin P$. Now $x^n t \in A$ implies $x^n t^n = (xt)^n \in A$, hence $xt \in \sqrt{A}$. Since $t \notin P$, we have $x \in (\sqrt{A})(P)$. Conversely, suppose $x \in (\sqrt{A})(P)$. Then there exists $t \notin P$ such that $xt \in \sqrt{A}$. So there exists a positive integer n such that $(xt)^n = x^n t^n \in A$. Now $t \notin P$ implies $t^n \notin P$, hence $x^n \in A(P)$, so $x \in \sqrt{A(P)}$.

3. Associated Primes in the Sense of Noether

We first introduce some terminology. The set complement of $Z(A)$ is a multiplicatively closed set (collection of all elements relatively prime to A), hence by Zorn's Lemma there are prime ideals containing A, contained in $Z(A)$, and maximal with respect to these properties. These prime ideals are called maximal prime divisors of A and denoted by *MxPD*. For discussions of *MxPD*, see [2], [3], and [5]. Furthermore, we denote the minimal prime overideal property by *MnPD*, i.e., P is called a *MnPD* of A if P is a minimal prime overideal of A.

We now list the various definitions of associated prime divisors as discussed in [1], [2], and [5]. Let P be a prime ideal containing A. Then:

(B) P is an associated prime divisor of A in the Bourbaki sense if $P = A:x$ for some $x \in R$.

(Z-S) P is an associated prime divisor of A in the Zarski-Samuel sense if $A:x$ is P-primary for some $x \in R$.

(Bw) P is an associated prime divisor of A in the weak Bourbaki sense if P is a *MnPD* of $A:x$ for some $x \in R$.

(K) P is an associated prime divisor of A in the Krull sense if $P = Z(A(P))$.

(N) P is an associated prime divisor of A in the Nagata sense if PR_S is a *MxPD* of AR_S for some multiplicatively closed set S.

Notation. If P is a prime containing an ideal A, then P is called a (B)-prime of A, when P is an associated prime divisor of A in the Bourbaki sense. Similarly for the other conditions.

It is known that in Noetherian rings these conditions are equivalent, that in general they are distinct conditions, and that they form an increasing sequence of implications (cf. [2] and [5]). We now state and prove the result of this section which places the (Ne)-primes between the (B)-primes and (Bw)-primes, shows that in general the (Ne)-primes are distinct, and neither imply nor are implied by the (Z-S)-primes. More precisely we have

PROPOSITION 5. *Let P be a prime ideal containing an ideal A. Then:*

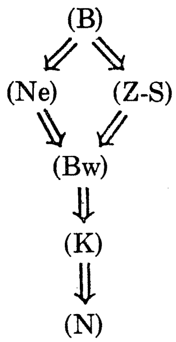
(**a**) *P a (B)-prime of A $\Longrightarrow$ P a (Ne)-prime of A, but not conversely.*

(**b**) *P a (Ne)-prime of A $\Longrightarrow$ P a (Bw)-prime of A, but not conversely.*

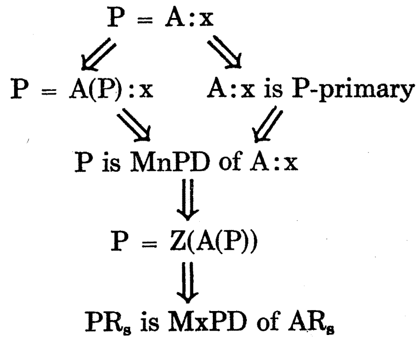
(**c**) *P a (Ne)-prime of A $\not\Longrightarrow$ P a (Z-S)-prime of A.*

(**d**) *P a (Z-S)-prime of A $\not\Longrightarrow$ P a (Ne)-prime of A.*

Before we prove proposition 5, it may be instructive to visualize the six associated prime divisor conditions via a diagram. In the diagram below, none of the implications can be reversed.



In terms of definitions the diagram is



We now prove proposition 5.

(**a**) Let P be a (B)-prime of A. Then there exists x such that $P = A{:}x$. Taking the P-component of both sides of the last equation we have $P = P(P) = (A{:}x)(P)$. Now by proposition 4, $(A{:}x)(P) = A(P){:}x$, hence $P = A(P){:}x$. Example 2, below, shows that the converse is not true.

(**b**) Suppose P is a (Ne)-prime of A, then there exists $x \in R$ such that $A(P){:}x = P$. But $A(P){:}x = (A{:}x)(P)$, and $(A{:}x)(P) = P \Longrightarrow P$ is a *MnPD* of $A{:}x$. Example 1 below, shows that $(Z\text{-}S) \not\Longrightarrow (Ne)$. Furthermore, since it is true that $(Z\text{-}S) \Longrightarrow (Bw)$, it follows that $(Bw) \not\Longrightarrow (Ne)$.

(**c**) Example 2, below.

(**d**) Example 1, below.

Example 1. This example is discussed by D. Underwood in [5, p. 74]. Let R be a rank one valuation ring with the additive group of real numbers as value group. Let $P = \{x \in R \mid v(x) > 0\}$ and $A = \{x \in R \mid v(x) \geq 1\}$. Then P is the unique proper prime of R, and so A is P-primary, hence $A(P) = A$. For any $x \in R\backslash A$, $A{:}x$ is P-primary, hence P is a $(Z\text{-}S)$-prime of A. Furthermore, in [5] it is shown that $A{:}x \neq P$ for any $x \in R$. Then since $A(P) = A$, it follows that $A(P){:}x \neq P$ for any $h \in R$. Thus, P is not a (Ne)-prime of A.

Example 2. Let K denote the field of two elements $\{0, 1\}$ and R the collection of all sequences on K which have "constant tails", i.e., $R = \{a = \{a_i\} \mid a_i \in K$ and there exists positive integer n such that for all positive integers p, $a_n = a_{n+p}\}$. Define addition and multiplication in R componentwise, i.e., $a + b = \{a_i + b_i\}$ and $ab = \{a_ib_i\}$. It is seen that R is a commutative ring with identity $e = \{a_i\}$ where $a_i = 1$ for all i. For each positive integer j, let $u_j = \{a_i\} \in R$ such that $a_i = 0$ for $i \neq j$ and $a_j = 1$. Also, let $v_j = \{a_i\} \in R$ such that $a_i = 0$ for $i < j$ and $a_i = 1$ for $i \geq j$. Then for each $x \in R$, there exists a finite set of positive integers, J, (possibly empty) and possibly an integer $p > \max_J \{j\}$ such that $x = v_p + \Sigma_J u_j$. Let $P = (\{u_i\}) = \{a \in R \mid$ there exists a nonnegative integer n with $a_n = a_{n+p} = 0$ for all positive integers $p\}$. Then P is a prime ideal. Let A denote the zero ideal in R. We now claim (a) P is a (Ne)-prime of A, (b) P is not a (B)-prime of A, and (c) P is not a $(Z\text{-}S)$-prime of A.

(**a**) $A(P) = \{y \in R \mid \text{there exists } x \notin P \text{ such that } xy = 0\}$. Let $y \in P$. Then there exists finite set of positive integers J, such that $y = \Sigma_J u_i$. Let $i - 1 = \max_J \{j\}$. Then $v_i \notin P$. Furthermore, since $i > j$ for each $j \in J$, it is seen that $v_i y = v_i(\Sigma_J u_j) = 0$. This means that $y \in A(P)$, and since y was any element of P, it follows that $P \subset A(P)$. But since the other inclusion is always true, $P = A(P)$. Now $P = P{:}1 = A(P){:}1$, hence P is a (Ne)-prime of A.

(**b**) Let $0 \neq x \in R$. Then $x = v_i + \Sigma_J u_j$ where J is a finite set of positive integers (possibly empty) and i, if it exists, is greater than $\max_J \{j\}$. If $J = \varnothing$, then i exists and so $u_i x \neq 0$. In this case $A{:}x \neq P$. Now suppose $J \neq \varnothing$ and let $j \in J$. Then again $u_j \in P$ and $xu_j = u_j \neq 0$, so $A{:}x \neq P$. Thus, since $A{:}x \neq P$ for each non zero element of R, it follows that P is not a (B)-prime of A.

(**c**) R is a Boolean ring and so the only primary ideals are the primes ideals themselves. Thus by part (b), $A{:}x$ is not P-primary for any $x \in R$, hence P is not a $(Z\text{-}S)$-prime of A.

4. Sufficient Conditions

In this final section we show that the sets of finitely generated primes are the same for the four types of associated prime divisors, (B), (Ne), (Bw), and $(Z\text{-}S)$. To this end we first record the following results:

PROPOSITION 6. *Suppose Q is P-primary and P is finitely generated. Then $Q{:}P > Q$.*

Proof. Let $P = (x_1, \cdots, x_n)$. If $P = Q$, there is nothing to show, for then $Q{:}P = R$. Hence assume $x_1, \cdots, x_p$ do not belong to Q, where $1 \leq p \leq n$. Now choose non negative integers n_i, $i = 1, \cdots, p$ such that $x = \Pi x_i^{n_i} \notin Q$, while $x_j x \in Q$ for each $j = 1, \cdots, n$. Then it is clear that $Q{:}P > Q$, since $x \in Q{:}P$.

COROLLARY. *Let P be a finitely generated MnPD of A. Then P is a (Ne)-prime of A.*

Proof. If P is a *MnPD* of A, then $A(P)$ is P-primary. Now the corollary follows from proposition 6 and condition (c) of proposition 1.

PROPOSITION 7. *Let A be an ideal. Then $\{$finitely generated (Bw)-primes of $A\}$ = $\{$finitely generated (Ne)-primes of $A\}$ = $\{$finitely generated (B)-primes of $A\}$ = $\{$finitely generated $(Z\text{-}S)$-primes of $A\}$.*

Proof. Because of the implications displayed in the diagram of proposition 5, it is sufficient to prove (1) finitely generated (Bw)-primes of A are (Ne)-primes of A, and (2) finitely generated (Ne)-primes of A are (B)-primes of A.

To prove (1), assume P is a finitely generated (Bw)-prime of A. Then there exists $x \in R$ such that $A(P){:}x$ is P-primary. Since P is finitely generated, $(A(P){:}x){:}P > A(P){:}x$, by proposition 6. Now let $y \in (A(P){:}x){:}P \backslash A(P){:}x$. Then yP is contained in $A(P){:}x$, which means $xyP \subset A(P)$, and so $xy \in$

$A(P):P$. But $y \notin A(P):x \Rightarrow xy \notin A(P)$. Hence $xy \in A(P):P\backslash A(P)$, i.e., $A(P):P > A(P)$, and so by condition (c) of proposition 1, P is a (Ne)-prime of A.

Now to prove (2), let $P = (x_1, \cdots, x_n)$ and let $A(P):y = P$. Now for each x_i there exists $u_i \notin P$ such that $(x_iu_i)y \in A$. Let $u = \Pi_1^n u_i$. Then $u \notin P$ and $ux_iy = \Pi_j u_j x_i y \in A$. Furthermore, $uy \notin A$, for if so, then $y \in A(P)$, whence $A(P):y \neq P$. We now claim $A:uy = P$. It follows that $P \subset A:uy$ since for each i, $x_i \in A:uy$ and $P = (x_1, \cdots, x_n)$.

Conversely, suppose $vuy \in A$. Then $vy \in A(P)$, as $u \notin P$. Hence $v \in A(P):y = P$, thus $P = A:uy$.

Corollary. *Let P be a finitely generated MnPD of A. Then P is an associated prime divisor of A in all six senses.*

Proof. If P is a *MnPD* of A, then P is a (Bw)-prime of A. Now a finitely generated (Bw)-prime of A is also a (B)-prime of A, hence corollary follows.

Corollary. *Let Q be P-primary and let P be finitely generated. Then P is an associated prime divisor of Q in all six senses.*

Proof. Clear.

Corollary. *Let $A = \bigcap_1^n Q_i$ be a normal primary decomposition and let $P = P_1$ be finitely generated. Then P is a (B)-prime of A. (Hence the finitely generated associated primes in a normal primary decomposition are associated prime divisors in all six senses.)*

Proof. The associated primes of a normal primary decomposition are (Bw)-primes.

Remark. Without the finitely generated conditions on P, the above corollaries are no longer true, cf. Example 1.

We end our discussion with the observation that if A has a normal primary decomposition, then $\{(B)\text{-primes of } A\} = \{(Ne)\text{-primes of } A\}$. To see this, first let us recall: if Q is P-primary, then $Q:x = R$, if $x \in Q$, $Q:x = Q$, if $x \notin P$, and $Q:x = Q*$, if $x \in P\backslash Q$ and where $Q*$ is P-primary.

Proposition 8. *Let $A = \bigcap_1^n Q_i$ be a normal primary decomposition for A, and suppose $P = P_1$ is a (Ne)-prime of A. Then P is a (B)-prime of A.*

Proof. Let the associated primes of A be indexed such that $P_i \subset P_1 = P$ for $i = 1, \cdots, k$, and if $k \neq n$, then $P_i \not\subset P_1$ for $k < i \leq n$. Now since P is a (Ne)-prime of A, there exists $x \in R$ such that $P = A(P):x = (\bigcap_1^k Q_i):x = \bigcap_1^k (Q_i:x)$. If $k = n$, we are done, for in this case $A(P) = A$, and so $A:x = P$. So assume $k < n$. Then $A:x = (\bigcap_1^n Q_i):x = \bigcap_1^k (Q_i:x) \cap (\bigcap_{k+1}^n Q_i) = P \cap (\bigcap_{k+1}^n Q_i:x)$. Now since $P_r \not\subset P$, for $r = k + 1, \cdots, n$, there exists $y_r \in Q_r\backslash P$. Let $y = \Pi_{k+1}^n y_r$. Then $y \in (\bigcap_{k+1}^n Q_j)\backslash P$. It then follows that $A:xy = (A:x):y =$

$P:y \cap (\cap_{k+1}{}^n(Q_1:x):y = P \cap R = P$, since $P:y = P$ and $(Q_1:x):y = R$ for $i = k + 1, \cdots, n$. Thus P is a (B)-prime of A.

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