VECTOR FIELDS ON MANIFOLDS WITH BOUNDARY

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Introduction

If M is a manifold with boundary, then a vector field on ∂M will be a map $v: \partial M \to TM$ such that $\pi \cdot v = 1$ where π is the projection in the tangent bundle of M. We will show that if M is compact, oriented, connected, $2n + 1$-dimensional and ∂M is connected, then a nowhere zero vector field v extends to a nowhere zero vector field on M if and only if the Euler class of the $2n$-plane bundle over ∂M of vectors normal to v is 0. This is a generalization of Hopf's classical result. The proof will depend on relative vector bundles. Most of the results about relative vector bundles are analogues of the results about standard vector bundles.

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Throughout the paper we follow the following conventions: all spaces are paracompact; all manifolds are compact, connected, oriented and equipped with a Riemannian metric; a map is a continuous function; and all cohomology groups are over Z unless otherwise indicated. Given an n-plane bundle γ we will sometimes, by an abuse of language, use γ to denote its total space. In any pair (B, B'), B' is *closed* unless we explicitly say otherwise.

I. Relative Plane Bundles

Definition 1.1. A *relative* (n, k)*-plane bundle*, Γ, is a pair (γ, γ') where $\gamma = (E, p, B)$ is an n-plane bundle and $\gamma' = (E', p', B')$ is a k-plane sub-bundle of γ. We will call (B, B') the base of Γ and say that Γ is a relative (n, k)-plane bundle over (B, B').

A *morphism* between (n, k)-plane bundles $\Gamma = (\gamma, \gamma')$ and $\Delta = (\delta, \delta')$ is a vector bundle morphism from γ to δ which restricts to a vector bundle morphism from γ' to δ'. If (B, B') is the base of Γ and Δ, then a (B, B')*-morphism* from Γ to Δ is a morphism which covers the identity map on (B, B'). If Γ and Δ are (B, B')-isomorphic, we will write $\Gamma = \Delta$. Using the standard result about vector bundles that a one-to-one B-morphism between n-plane bundles is a B-isomorphism, we get the analogous result for relative bundles.

Proposition 1.2. *Suppose* (γ, γ') *and* (δ, δ') *are* (n, k)*-plane bundles over* (B, B') *and* $f: (\gamma, \gamma') \to (\delta, \delta')$ *is a* (B, B')*-morphism. If* f *is one-to-one, then* f *is a* (B, B')*-isomorphism.*

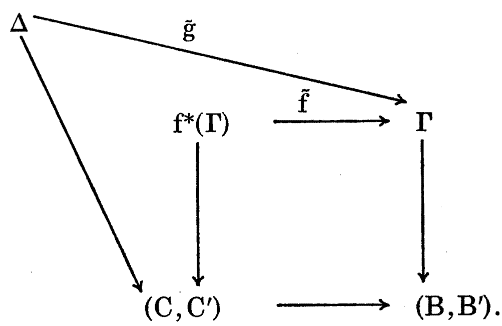
We will omit the proof of any result such as the above which is essentially similar to the corresponding non-relative one [cf. 3].

If $\Gamma = (\gamma, \gamma')$ is a relative (n, k)-plane bundle over (B, B') and $f: (C, C') \to (B, B')$, then $(f^*(\gamma), f\,|\,_{C'}{}^*(\gamma'))$ is the *pullback* of Γ by f. We will denote the pullback by $f^*(\Gamma)$ or $f^*(\gamma, \gamma')$. Obviously, it is a relative (n, k)-plane bundle.

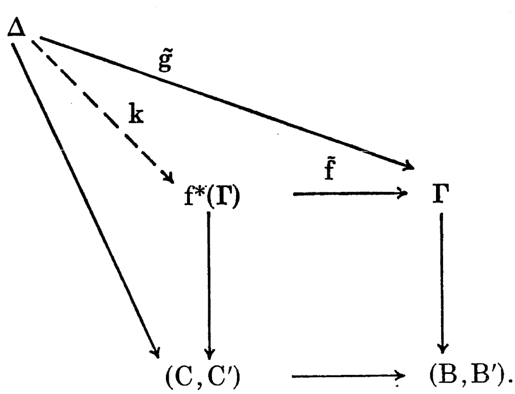
PROPOSITION 1.3. *Suppose $f: (C, C') \to (B, B')$ and $\Gamma = (\gamma, \gamma')$ is a relative (n, k)-plane bundle over (B, B'). Then we have a morphism $\tilde{f}: f^*(\Gamma) \to \Gamma$ and a commutative diagram*

$$\begin{array}{ccc} f^*(\Gamma) & \xrightarrow{\tilde{f}} & \Gamma \\ \downarrow & & \downarrow \\ (C,C') & \xrightarrow{f} & (B,B') \end{array}$$

where the vertical maps are projections and $\tilde{f}$ is one-to-one on each fibre. Suppose $\Delta = (\delta, \delta')$ is also an (n, k)-plane bundle over (C, C') and that we have a morphism $\tilde{g}$ such that



Then there is a morphism k such that we have the following commutative diagram



Furthermore if $\tilde{g}$ is one-to-one on each fibre, then k is a (C, C')-isomorphism from Δ to $f^*(\Gamma)$.

Note that in the above proposition we use, by an abuse of notation, the same symbol for the total space of a relative bundle as for the bundle itself. We will continue to do this when convenient.

Let θ^n be the trivial n-plane bundle over B, θ^k the trivial k-plane bundle over $B' \subset B$ considered as a sub-bundle of θ^n in the natural way. Then (γ, γ'), a relative (n, k)-plane bundle over (B, B'), is *trivial* if and only if $(\gamma, \gamma') = (\theta^n, \theta^k)$. If $(C, C') \subset (B, B')$, then the *restriction* of a relative bundle $\Gamma = (\gamma, \gamma')$ over (B, B') to (C, C') is $(\gamma \mid_C, \gamma' \mid_{C'})$. We will denote it by $(\gamma, \gamma') \mid_{(C,C')}$ or $\Gamma \mid_{(C,C')}$. We now note some technical results which we will need later.

PROPOSITION 1.4. *Let* $\Gamma = (\gamma, \gamma')$ *be a relative* (n, k)*-plane bundle over* $(B, B) \times I$. *Then there exists an open cover* $\{U_j\}$, $j \in J$, *of* B *such that* $\Gamma \mid_{(U_j, U_j) \times I}$ *is trivial.*

By Proposition 1.3 and Lemma 1.4.2 of [1] we get

PROPOSITION 1.5. *Let* (γ, γ') *be a relative* (n, k)*-plane bundle over* (B, B'). *Suppose* B' *has a neighborhood* B'' *of which it is a retract. Then there exists a neighborhood of* B', B''', *a* k*-plane bundle,* γ'', *over* B''' *and an* $(n\text{-}k)$*-plane bundle* γ''' *over* B''' *such that:*

1) $\gamma'' \mid_{B'} = \gamma'$;
2) γ'' *and* γ''' *are sub-bundles of* γ;
3) $(\gamma'' \oplus \gamma''', \gamma'') = (\gamma \mid_{B'''}, \gamma'')$.

Furthermore, if γ *and* γ' *are oriented, so are* γ'' *and* γ'''.

PROPOSITION 1.6. *Let* $\Gamma = (\gamma, \gamma')$ *be a relative* (n, k)*-plane bundle over* $(B, B') \times I$. *Suppose* B' *has a neighborhood of which it is a retract and define* $r: B \times I \to B \times I$ *by* $r(b, t) = (b, 1)$. *Then there exists a morphism* $f: \Gamma \to \Gamma \mid_{(B,B') \times [1]}$ *covering* r *such that* f *restricted to a fibre is one-to-one.*

In order to prove Proposition 1.6 we will need the following:

LEMMA 1.7. *Suppose the conditions of Proposition* 1.6 *are satisfied. Then there exists an open set* $B'' \supset B'$, *a* k*-plane bundle* γ'' *over* $B'' \times I$ *and an* $(n\text{-}k)$*-plane bundle* γ''' *over* $B'' \times I$ *such that*

1) $\gamma'' \mid_{B' \times I} = \gamma'$
2) γ'' *and* γ''' *are sub-bundles of* γ
3) $(\gamma''' \oplus \gamma'', \gamma'') = (\gamma \mid_{B''}, \gamma'')$.

Furthermore, there exists a locally finite open cover of B, $\{U_j\}$, $j \in J$, *with a sub-collection* $\{U_j\}$, $j \in J'$, *covering* B'' *such that*

$$j \in J' \Rightarrow U_j \subset B'' \text{ and } (\gamma, \gamma'') \mid_{(U_j, U_j) \times I} \text{ is trivial,}$$
$$j \in J - J' \Rightarrow U_j \subset B - B' \text{ and } \gamma \mid_{U_j \times I} \text{ is trivial.}$$

The proof of Proposition 1.6 is now analogous to the non-relative case [3]. Using Proposition 1.6 we get

THEOREM 1.8. *Suppose* $B' \subset B$ *has some neighborhood of which it is a retract.*

If $f_0, f_1: (B, B') \to (C, C')$ are homotopic and $\Gamma = (\gamma, \gamma')$ is a relative (n, k)-plane bundle over (C, C'); then $f_0^*(\Gamma) = f_1^*(\Gamma)$.

We now want a universal (n, k)-plane bundle. Let $\boldsymbol{R}^\infty = \oplus_{i=1}^\infty \boldsymbol{R}$. Let G_n be the Grassman manifold of oriented n-planes in $\boldsymbol{R}^\infty$. Let $\gamma_n = (E_n, p_n, G_n)$ be a natural n-plane bundle over G_n. (If $p \in G_n$, then a point in the fibre above p is a pair (v, p) such that $v \in p$.)

PROPOSITION 1.9. *There are natural inclusions, ι of $G_k \times G_{n-k}$ into G_n, and κ of G_k into G_n.*

From here on we will consider $G_k \times G_{n-k}$ and G_k to be the subsets of G_n given by the above result. Let γ_k' be the (oriented) obvious k-plane bundle over $G_k \times G_{n-k} \subset G_n$ such that if $(p, q) \in G_k \times G_{n-k}$ then the fibre above (p, q) is the set of all vectors in p. Then (γ_n, γ_k') is a relative (n, k)-plane bundle.

Note that $\gamma_k' \mid_{G_k} = \gamma_k$. Therefore we will consider γ_k to be this sub-bundle of γ_n. Then we get the following

PROPOSITION 1.10. *Let i be the inclusion of (G_n, G_k) in $(G_n, G_k \times G_{n-k})$. Then $i^*(\gamma_n, \gamma_k') = (\gamma_n, \gamma_k)$.*

PROPOSITION 1.11. *Suppose (γ, γ') is a relative (n, k)-plane bundle over (B, B'), $\gamma = (E, p, B)$, and $\hat{f}: E \to \boldsymbol{R}^\infty$ is linear and one-to-one on each fibre. If $\{\hat{f}(v) \mid v \in p^{-1}(b), b \in B'\} \in G_k \times G_{n-k} \subset G_n$ for all $b \in B'$, then $\hat{f}$ induces $\tilde{f}$, f such that we have the following commutative diagram:*

$$\begin{array}{ccc} (E, E') & \xrightarrow{\tilde{f}} & (\gamma_n, \gamma_k') \\ \downarrow & & \downarrow \\ (B, B') & \xrightarrow{f} & (G_n, G_k \times G_{n-k}). \end{array}$$

THEOREM 1.12. *Let $f, g: (B, B') \to (G_n, G_k \times G_{n-k})$ be maps such that $f^*(\gamma_n, \gamma_k') = g^*(\gamma_n, \gamma_k')$. Then $f \simeq g$.*

The proof follows from

LEMMA 1.13. *We can define homotopies $\hat{g}^a, \hat{g}^b: \boldsymbol{R}^\infty \times I \to \boldsymbol{R}^\infty$ which induce maps $\tilde{g}^a, \tilde{g}^b: (\gamma_n, \gamma_k') \times I \to (\gamma_n, \gamma_k')$. These induce $g^a, g^b: (G_n, G_k \times G_{n-k}) \times I \to (G_n, G_k \times G_{n-k})$. Furthermore g_0^a and g_0^b are the identity.*

We now note that "any" oriented (n, k)-plane bundle is the pullback of (γ_n, γ_k').

Definition 1.14. An (n, k)-plane bundle (γ, γ') is oriented if γ and γ' are.

THEOREM 1.15. *Let (γ, γ') be an oriented (n, k)-plane bundle over (B, B') where B' has a neighborhood of which it is a retract. Then there exists a map $f: (B, B') \to (G_n, G_k \times G_{n-k})$ such that $f^*(\gamma_n, \gamma_k') = (\gamma, \gamma')$. If $(\gamma \mid_{B'}, \gamma') = (\gamma' \oplus \theta^{n-k}, \gamma')$ where θ^{n-k} is the trivial bundle over B', f can be chosen so that $f(B') \subset G_k$.*

The proof follows from Propositions 1.5 and 1.11.

Using the above results, we get the following:

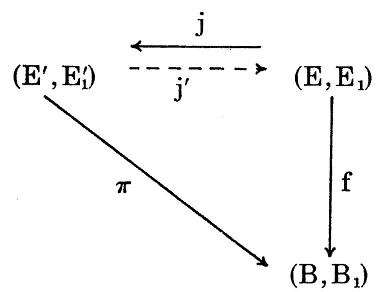
THEOREM 1.16. *If B' has a neighborhood of which it is a retract, there is a one-to-one correspondence between (isomorphism classes of) oriented relative (n, k)-plane bundles over (B, B') and $[(B, B'); (G_n, G_k \times G_{n-k})]$.*

Proof: Theorems 1.8, 1.12, 1.15.

II. A Fibration

In this section we will consider a certain "relative" fibration which we will need. First we prove a relative version of the standard result that "up to homotopy" every map is a fibration.

PROPOSITION 2.1. *Let $f:(E, E_1) \to (B, B_1)$ be a map such that $f \mid_{E_1}$ is a homeomorphism onto B_1. Then there exists a pair (E', E_1') and maps $\pi:(E', E_1') \to (B, B_1)$, $j:(E, E_1) \to (E', E_1')$, $j':(E', E_1') \to (E, E_1)$ such that π is a fibration, $\pi \mid_{E_1'}: E_1' \to B_1$ is a homeomorphism and j and j' are homotopy equivalences. Furthermore, the following diagram homotopy commutes (as maps of pairs):*



(the triangle involving j is strictly commutative).

Proof: Let $E' = \{(e, \lambda) \in E \times B^I \mid f(e) = \lambda(0)\}$ and $E_1' = \{(e, f(e)^*) \in E' \mid f(e)^*$ is the constant path at $f(e)$ and $e \in E_1\}$. Define $j:(E, E_1) \to (E', E_1')$ by $j(e) = (e, f(e)^*)$, $j':(E', E_1') \to (E, E_1)$ by $j'(e, \lambda) = e$ and $\pi:(E', E_1') \to (B, B_1)$ by $\pi(e, \lambda) = \lambda(1)$. Note that $j'j$ is the identity. jj' is homotopic to the identity by the homotopy $h:(E', E_1') \times I \to (E', E_1')$ which retracts all paths to their initial points: $h_t(e, \lambda) = (e, \lambda_t)$ where $\lambda_t: I \to B$ is defined by $\lambda_t(s) = \lambda(ts)$. So $h_0 = jj'$ and $h_1 = 1_{(E', E_1')}$. π is a fibration by [7, p. 84, Proposition 1].

Since the inclusion $G_{n-1} \to G_n$ has fibre S^{n-1} we get

PROPOSITION 2.2. *Let $i:(G_{n-1}, G_{n-1}) \to (G_n, G_{n-1})$ be the inclusion map. Let $\pi: G_{n-1}' \to G_n$ be the fibration given by Proposition 2.1 applied to i. Then the fibre of π has the homotopy type of S^{n-1}.*

PROPOSITION 2.3. *Let $\pi: G_{2n}' \to G_{2n+1}$ be the fibration in Proposition 2.2. Let ι be a generator of $H^{2n}(S^{2n})$. Then $\tau(\iota) = W_{2n+1}$ where $W_{2n+1} \in H^{2n+1}(G_{2n+1})$ is the integral Stiefel Whitney class in dimension $2n + 1$.*

Proof: [5, Theorem 6.16].

PROPOSITION 2.4. *$H^{2n+1}(G_{2n+1}, G_{2n}) = Z$ and we can choose a generator of it, ε, such that $\delta e = 2\varepsilon$, where $e \in H^{2n}(G_{2n})$ is the Euler class and δ is the boundary map in the cohomology sequence of the pair (G_{2n+1}, G_{2n}). Furthermore, $i^*(\varepsilon) = W_{2n+1}$ where $W_{2n+1} \in H^{2n+1}(G_{2n+1})$ is the integral Stiefel Whitney class in dimension $2n + 1$ and i is the inclusion map of G_{2n+1} in (G_{2n+1}, G_{2n}).*

Proof: Immediate using the well known fact that (G_{2n+1}, G_{2n}) is the Thom space of γ_{2n+1} over G_{2n+1}.

Following Kervaire, who defined relative Stiefel Whitney classes [4], we will call ε the relative Euler class. Using the above Propositions and noting that W_{2n+1} is the first Postnikov invariant for the inclusion $G_{2n} \to G_{2n+1}$ we get

LEMMA 2.5. *Let M be a $2n + 1$-dimensional manifold with boundary. Suppose $f: (M, \partial M) \to (G_{2n+1}, G_{2n})$ and $f^*(\varepsilon) = 0$, where ε is the generator of $H^{2n+1}(G_{2n+1}, G_{2n}) = Z$ given in Proposition 2.4. Then f is homotopic (as a map of pairs) to some f' where $f'(M) \subset G_{2n}$.*

III. The Main Theorem

If M is a manifold with boundary, a vector field on ∂M is a map $v: \partial M \to TM$ such that $\pi v = 1_{\partial M}$ where π is the projection for the tangent bundle of M. Note that we don't require a vector field on ∂M to lie in $T(\partial M)$.

Definition 3.1. Let M^n be a manifold and S a subset of M. Given a nowhere zero vector field on S, v, let ν_v be the sub-bundle of $TM \mid_S$ of vectors normal to v. More explicitly, $(\nu_v)_x = \{u \in TM_x \mid u \text{ is normal to } v(x)\}$. ν_v has a natural orientation since TM is oriented (i.e., if $u_1, \cdots, u_{n-1}$ form a basis for $(\nu_v)_x$, then they give the right orientation of it, if and only if $u_1, \cdots, u_{n-1}, v(x)$ give the orientation of TM_x).

We can now state and prove our main result.

THEOREM 3.2. *Let M be a $2n + 1$-dimensional manifold with boundary and v a nowhere zero vector field on ∂M. Then v can be extended to a nowhere zero vector field on M if and only if $e(\nu_v) = 0$, where e denotes the Euler characteristic class.*

Proof: By Theorem 1.15 we can find a map $f: (M, \partial M) \to (G_{2n+1}, G_{2n})$ such that $f^*(\gamma_{2n+1}, \gamma_{2n}) = (TM, \nu_v)$. Since $\delta e = 2\varepsilon$ by Proposition 2.4, we get

$$\begin{array}{ccc}
H^{2n}(G_{2n}) & \longrightarrow & H^{2n+1}(G_{2n+1}, G_{2n}) \\
 & e \longrightarrow 2\varepsilon & \\
f^* \Big\downarrow & \Big\downarrow \qquad \Big\downarrow & \Big\downarrow f^* \\
 & e(\nu_v) \underset{\delta}{\longrightarrow} 2f^*(\varepsilon) & \\
H^{2n}(\partial M) & \longrightarrow & H^{2n+1}(M, \partial M)
\end{array}$$

where the rows come from the exact sequences of the pairs involved. We know that $f^*(e) = e(\nu_v)$ be the definition of the Euler class of ν_v. Since $H^{2n+1}(M) = 0$ and $H^{2n}(\partial M) = H^{2n+1}(M, \partial M) = Z$, $\delta: H^{2n}(\partial M) \to H^{2n+1}(M, \partial M)$ is an isomorphism. So $f^*(\varepsilon) = 0$ if and only if $e(\nu_v) = f^*(e) = 0$.

We now split the proof:

Only if: Suppose v extends to a nowhere zero vector field on M, **v**. Let θ be the trivial line bundle over M generated by **v**. Then $(TM, \nu_v) = (\nu_{\mathbf{v}} \oplus \theta, \nu_{\mathbf{v}})$. Take $g: M \to G_{2n}$ so that $g^*(\gamma_{2n}) = v_{\mathbf{v}}$. Then considering g as a map from $(M, \partial M)$ to (G_{2n+1}, G_{2n}), we get

$$\begin{aligned} g^*(\gamma_{2n+1}, \gamma_{2n}) &= (g^*(\gamma_{2n+1} \mid G_{2n}), g^*(\gamma_{2n})) \\ &= (g^*(\gamma_{2n} \oplus \theta), g^*(\gamma_{2n})) \\ &= (g^*(\gamma_{2n}) \oplus \theta, g^*(\gamma_{2n})) \\ &= (\nu_{\mathbf{v}} \oplus \theta, \nu_{\mathbf{v}}) \\ &= (TM, \nu_{\mathbf{v}}). \end{aligned}$$

Since $f^*(\gamma_{2n+1}, \gamma_{2n}) = (\mathrm{TM}, \nu_v) = g^*(\gamma_{2n+1}, \gamma_{2n})$ and $(\gamma_{2n+1}, \gamma_{2n}) = i^*(\gamma_{2n+1}, \gamma'_{2n})$ by Proposition 1.10 where $i: (G_{2n+1}, G_{2n}) \to (G_{2n+1}, G_{2n} \times G_1)$ we see that $(if)^*(\gamma_{2n+1}, \gamma'_{2n}) = (ig)^* (\gamma_{2n+1}, \gamma'_{2n})$. By Theorem 1.10 $if \simeq ig: (M, \partial M) \to (G_{2n+1}, G_{2n} \times G_1)$. But we can factor $ig: (M, \partial M) \to (G_{2n+1}, G_{2n} \times G_1)$ through (G_{2n}, G_{2n}) since $g(M) \subset G_{2n}$. Thus we get the following commutative diagram

$$\begin{array}{ccc} H^{2n+1}(G_{2n+1}, G_{2n} \times G_1) & \xrightarrow{(ig)^*} & H^{2n+1}(M, \partial M) \\ i^* \downarrow & & \uparrow g^* \\ H^{2n+1}(G_{2n+1}, G_{2n}) & \xrightarrow{\text{inclusion}^*} & H^{2n+1}(G_{2n}, G_{2n}) = 0 \end{array}$$

Since $\tilde{H}^*(G_1) = 0$, i^* is an isomorphism. Let $\varepsilon' = (i^*)^{-1}(\varepsilon)$. Then $(ig)^*(\varepsilon') = 0$. Therefore $0 = (if)^*(\varepsilon') = f^*i^*(\varepsilon') = f^*(\varepsilon)$. So $e(\nu_v) = 0$ since it is zero if $f^*(\varepsilon) = 0$.

If: Assume that $f^*(\varepsilon) = 0$. By Lemma 2.5, $f \simeq g$ where $g(M) \subset G_{2n}$. Then

$$\begin{aligned} (TM, \nu_v) &= f^*(\gamma_{2n+1}, \gamma_{2n}) \\ &= g^*(\gamma_{2n+1}, \gamma_{2n}) \\ &= (g^*(\gamma_{2n+1}), g^*(\gamma_{2n})) \\ &= (g^*(\gamma_{2n+1} \mid G_{2n}), g^*(\gamma_{2n})) \\ &= (g^*(\gamma_{2n} \oplus \theta), g^*(\gamma_{2n})) \\ &= (g^*(\gamma_{2n}) \oplus \theta, g^*(\gamma_{2n})). \end{aligned}$$

Let $\varphi: (TM, \nu_v) \to (g^*(\gamma_{2n}) \oplus \theta, g^*(\gamma_{2n}))$ be the relative bundle isomorphism. Let $u(x) \in TM_x$ be the unit vector normal to $\varphi^{-1}(g^*(\gamma_{2n}))$ with the "right" orientation. That is, if $u_1, \cdots, u_{2n} \in \varphi^{-1}(g^*(\gamma_{2n})) \mid_x$ is a basis giving the orientation of $\varphi^{-1}(g^*(\gamma_{2n})) \mid_x$, then $u(x), u_1, \cdots, u_{2n}$ is a basis of TM_x giving the orientation of TM_x. Take $d: M \to \mathbf{R}$ a nowhere zero differentiable function which agrees with $\| v \|$ on ∂M. This can be done by extending $\| v \|$ to a positive differentiable function on an open collar neighborhood, N, of ∂M and using a partition of unity subordinate to $\{M - \partial M, N\}$. Then $\boldsymbol{v}: M \to TM$, defined by $\boldsymbol{v}(x) = d(x)u(x)$, is a nowhere zero vector field on M. By construction, $\boldsymbol{v} \mid_{\partial M} = v$.

As a corollary to this theorem, we get the classical result of Hopf, in the case where M is oriented and odd-dimensional.

COROLLARY 3.3. *Let M be an odd-dimensional manifold with boundary. Let $v: \partial M \to TM$ be a nowhere zero vector field pointing out of M normal to ∂M (i.e., if $u \in T(\partial M)_x$, then $v(x)$ is normal to u). Then v extends to a nowhere zero vector field on M if and only if $\chi(M) = 0$.*

Proof: Since v is normal to $T(\partial M)$, ν_v, which is the bundle over ∂M of vectors normal to v, is $T(\partial M)$. So $e(\nu_v) = e(T(\partial M)) = \chi(\partial M)\iota = 2\chi(M)\iota$.

We also get the following result which is *not* true if N is odd-dimensional (cf. $M_1 = M_2 = D^2$ the unit disc).

COROLLARY 3.4. *Suppose $N = \partial M_1 = \partial M_2$ is a $2n$-dimensional manifold and let M be the differentiable manifold formed by $M_1 \cup M_2$. Let $v: N \to TM \mid_N = TM_1 \mid_{\partial M_1} = TM_2 \mid_{\partial M_2}$ be a nowhere zero vector field. Then v extends to a nowhere zero vector field on M_1 if and only if it extends to a nowhere zero vector field on M_2.*

Proof: Let ν_v be the vector bundle over $\partial M_1 = \partial M_2$ of vectors normal to v. Then by Theorem 3.2, v extends to a nowhere zero vector field on M_i if and only if $e(\nu_v) = 0$.

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