THE MAXIMAL ABELIAN EXTENSION OF A LOCAL FIELD AS A KUMMERIAN EXTENSION*

By Victor S. Albis González

§1. Introduction

We shall consider a local field L of characteristic 0, i.e., a finite extension of a p-adic number field, p being a rational prime; further, we assume that $q \geq 1$ represents the highest power of p for which the q-*th* roots of 1 are contained in L (they form a finite group). The following notations and results will be used consistently in this paper:

$n = [L; Q_p]$, where Q_p denotes the field of p-adic numbers.

$\mathcal{O}_L$ = *ring of integers of* L; if $L = Q_p$ we write Z_p instead of $\mathcal{O}_{Q_p}$.

$\mathcal{P}_L$ = *the maximal prime ideal of* $\mathcal{O}_L$.

ν_L = *the associated valuation*; any element $\pi \in \mathcal{O}_L$, such that $\nu_L(\pi) = 1$ is called a *uniformizing parameter* of L.

$U_L = \{\alpha \in \mathcal{O}_L ; \nu_L(\alpha) = 0\}$ = the *group of units of* $\mathcal{O}_L$

$U_{L,i} = 1 + \mathcal{P}_L^{\,i}$, $i = 1, 2, \cdots$; in particular, $1 + \mathcal{P}_L = U_{L,1}$ is called the *group of principal units of* L.

μ_q = the *group of q-th roots of* 1 (q defined as above) contained in L.

μ'_q = *the group of roots of* 1 *contained in L whose orders are prime to p.*

It is well-known [6: p. 78] that any element γ of $L^\times$ can be written uniquely as $\gamma = \pi^m\epsilon$, where π is a fixed uniformizing parameter of L, $m \in Z$ and $\epsilon \in U_L$. Further, a celebrated theorem of Hensel [6: p. 78] says that

$$U_L \approx \mu'_q \times \mu_q \times (\underbrace{Z_p \times \cdots \times Z_p}_{n\ times}). \tag{1}$$

where the groups on the right hand side of (1) are written additively. Hence

$$L^\times \approx Z \times \mu'_q \times \mu_q \times (\underbrace{Z_p \times \cdots \times Z_p}_{n\ times}) \tag{2}$$

Let us suppose now that $q > 1$, i.e., that the group μ_q is not trivial, and let us denote by L_a/L *the maximal abelian extension of* L. Its Galois group Gal (L_a/L) is given by

$$\hat{Z} \times \mu'_q \times \mu_q \times (\underbrace{Z_p \times \cdots \times Z_p}_{n\ times}),$$

where $\hat{Z} = \Pi_{all\ primes} Z_p$ is *the total completion of* Z[6: p. 81]. Denoting by L_q the fixed field of μ_q in L_a/L, it follows that L_a/L_q *is a (cyclic) kummerian extension of degree* q, and it is natural, in the spirit of class field theory, to ask which elements

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$\beta \in L^{\times}$ satisfy the condition $L_q(\sqrt[q]{\beta}) = L_a$. It will suffice to determine them $mod\ L^{\times q}$ (see proposition 1.)

We remark immediately the following proposition:

PROPOSITION 1. *To determine all $\beta \in L^{\times}$ mod $L^{\times q}$ satisfying $L_a = L_q(\sqrt[q]{\beta})$, it is necessary and sufficient to determine all kummerian extensions $L(\sqrt[q]{\beta})/L$, of degree q, which are linearly disjoint from L_q/L.*

The property: *$L(\sqrt[q]{\beta})/L$ and L_q/L are linearly disjoint,* will be called, from now on *property P*. Simetimes we shall say "β satisfies P", and some other times "$L(\sqrt[q]{\beta})/L$ satisfies P," hoping no confusion will arise from this.

We intend to transform condition P into a statement on the reciprocity law; in order to achieve this, it is desirable to have a convenient basis of $L^{\times}$ *mod* $L^{\times q}$ to simplify the computations with the *local symbols*. Let us recall what they are: If $L(\sqrt[q]{\beta})/L$ is a kummerian extension of degree $q > 1$, the local symbol (α,β) with respect to the *q-th* powers is defined by the relation

$$(3) \qquad (\alpha, L(\sqrt[q]{\beta})/L)(\sqrt[q]{\beta}) = (\alpha,\beta)\sqrt[q]{\beta}$$

where $(-, L(\sqrt[q]{\beta})/L): L^{\times} \to Gal\ (L(\sqrt[q]{\beta})/L)$ is the *reciprocity map*; (α,β) is a *q-th* root of 1, and it is independent of the choice of $\sqrt[q]{\beta}$ [4: p. 242]. The following two propositions contain useful information relating (α,β) to $(\alpha, L(\sqrt[q]{\beta})/L$. Here ζ_q is a fixed primitive *q-th* root of 1.

PROPOSITION 2. *Let $K = L(\sqrt[q]{\beta})$ be a kummerian extension of L of degree $q > 1$, and let $\tau \in Gal\ (K/L)$ be such that $\tau(\sqrt[q]{\beta}) = \zeta_q\sqrt[q]{\beta}$. If $(\alpha, L(\sqrt[q]{\beta})/L) = \tau^a$, then $(\alpha,\beta) = \zeta_q^a$.*

Proof: $\tau^a(\sqrt[q]{\beta}) = \zeta_q^a \cdot \sqrt[q]{\beta} = (\alpha,\beta)\sqrt[q]{\beta}$.

PROPOSITION 3. *Same notations as in proposition 1, and let $(\overline{\alpha,\beta})$ denote the local symbol with respect to the $\hat{q}$-th powers, $1 < \hat{q} \leq q$. Then $(\overline{\alpha,\beta}) = \zeta_{\hat{q}}^a$, where $\zeta_{\hat{q}} = \zeta_q^{q/\hat{q}}$.*

Proof: If τ_0 denotes the restriction of τ to $L(\sqrt[\hat{q}]{\beta})/L$, then τ_0 is a generator of $Gal(L(\sqrt[\hat{q}]{\beta})/L)$, and since $(\alpha, L(\sqrt[q]{\beta})/L\ |_{L(\sqrt[\hat{q}]{\beta})} = (\alpha, L(\sqrt[\hat{q}]{\beta})/L$, we get $(\alpha, L(\sqrt[\hat{q}]{\beta})/L) = \tau_0^a$; taking $\sqrt[\hat{q}]{\beta} = (\sqrt[q]{\beta})^{q/\hat{q}}$, we have thus $\tau_0(\sqrt[\hat{q}]{\beta}) = \zeta_q^{q/\hat{q}} \cdot \sqrt[\hat{q}]{\beta}$, from which $(\overline{\alpha,\beta}) = \zeta_{\hat{q}}^a$ follows.

§2. The arithmetic version of condition P

First we recall that for every natural number $m > 0$, there exists a unique unramified extension $L_{(m)}$ of degree m, contained in the maximal unramified extension L_{nr} of L; also, for any intermediate field $L \subseteq M \subseteq L_{nr}$, the Galois group of M/L is generated by the restriction of the Frobenius automorphism σ of L_{nr}/L to M/L (or, if preferred, $\sigma = \varprojlim \sigma\ |_M$), and we have, for M/L finite and unramified,

$$\sigma(\alpha) \equiv \alpha^{N(\mathfrak{P}_L)}\ (mod\ \mathfrak{P}_M), \quad \text{for all } \alpha \in \mathfrak{O}_M\ ,$$

where $N(\mathcal{P}_L)$ denotes the *absolute norm* of the ideal $\mathcal{P}_L$. Thus when mentioning the Frobenius automorphism of a particular intermediate field of L_{nr}/L (in particular, of the inertia field of any finite extension of L), we shall always be referring to the restriction of σ to that field. Also we shall always identify $Gal\,(L_{nr}/L)$ with $\hat{Z}$.

In the second place, we will use the results, due to KOCH, contained in the following two theorems:

THEOREM 1 [6: p. 98]. *Let* $n = [L : Q_p] \equiv 0 \pmod 2$, ζ_q *a primitive q-th root of* 1 *and* q' *the highest power of* P, *less than* q, *for which* $L(\sqrt[q']{/\zeta_q})/L$ *is unramified (i.e.,* $L(\sqrt[q']{\zeta_q})/L$ *is the inertia field of* $L(\sqrt[q]{\zeta_q})/L$. *Let* σ *denote the Frobenius automorphism of* $L(\sqrt[q]{\zeta_q})/L$, *and let* g *be a rational integer such that* $\sigma(\sqrt[q']{\zeta_q}) = (\sqrt[q']{\zeta_q})^g$. *Then*:

a) $g \equiv 1 \ (mod\ q)$
b) There is a basis $\{\pi, \alpha_0, \alpha_1, \cdots, \alpha_n\}$ *of* $L^\times$ *mod* $L^{\times q}$ *with the following properties*:
  (I) π *is a uniformizing parameter of* L;
  (II) α_0 *is q-primary, i.e.,* $L(\sqrt[q]{\alpha_0})/L$ *is unramified*;
  (III) $\alpha_1, \alpha_2, \cdots, \alpha_n \in U_L$;
  (IV) if $(-,-)$ *is the local symbol on L with respect to q-th powers, then*

$$(\pi, \alpha_0) = \zeta_q ;$$

$$(\alpha_{2\nu}, \alpha_{2\nu-1})^{-1} = (\alpha_{2\nu-1}, \alpha_{2\nu}) = \zeta_q \quad for \quad \nu = 1, \cdots, n/2,$$

and for all remaining pairs of distinct basic elements, the symbol equals 1;

  (V) $\sigma(\sqrt[q]{\alpha_0}) = \zeta_q \sqrt[q]{\alpha_0}$;
  (VI) $\zeta_q = \alpha_0^{(g-1)/q}\alpha_1^{q'}$ *mod* $L^{\times q}$, *and* $g - 1/q \not\equiv 0 \ (mod\ p)$ *if* $q' > 1$.

Moreover, if $q' = 1$ *we may take* $\alpha_1 = \zeta_q$ *and* $g = 1$.

COROLLARY. *Let* L/Q_p *satisfy the conditions of theorem* 1, *and let* $\{\pi, \alpha_0, \alpha_1, \cdots, \alpha_n\}$ *be the basis constructed there. Then*

$$(\pi, \zeta_q) = \zeta_q^{(g-1)/q}; \ (\alpha_2, \zeta_q) = \zeta_q^{-q'};$$

$$(\zeta_q, \zeta_q) = 1; \ (\alpha_\nu, \zeta_q) = 1 \quad for \quad \nu \neq 2.$$

Proof: From

$$\zeta_q = \alpha_0^{(g-1)/q}\alpha_1^{q'} \quad mod \quad L^{\times q}$$

we obtain $(\pi, \zeta_q) = \zeta_q^{(g-1)/q}$ and $(\alpha_2, \zeta_q) = (\alpha_2, \alpha_1)^{q'} = \zeta_q^{-q'}$. With the exception of $(\zeta_p, \zeta_q) = 1$ and $(\alpha_1, \zeta_q) = 1$, the remaining relations are trivially verified. Let us compute (ζ_q, ζ_q). If $p \neq 2$, we easily see that $(\zeta_q, \zeta_q) = 1$. If $p = 2$, $q \geq 4$, we have

$$1 = (-\zeta_q, \zeta_q) = (-1, \zeta_q)(\zeta_q, \zeta_q) = (\zeta_q, \zeta_q)^{1+q/2} = \zeta_q^{x(1+q/2)},$$

where we have used $\zeta_q^x = (\zeta_q, \zeta_q)$ and $-1 = \zeta_q^{q/2}$. Therefore

$$x(q/2+1) \equiv 0(mod\, q) \Rightarrow x \equiv 0(mod\, q) \Rightarrow (\zeta_q, \zeta_q) = 1,$$

since $1 + q/2$ is odd. If $p = q = 2$, we have

$$\begin{aligned}(-1, -1)\sqrt{-1} &= (-1, L(\sqrt{-1})/L)(\sqrt{-1}) \\ &= (N_{L/Q_2}(-1), Q_2(\sqrt{-1})/Q_2)\sqrt{-1} = \sqrt{-1},\end{aligned}$$

since $N_{L/Q_2}(-1) = (-1)^n = 1$, n being even; therefore $(-1, -1) = 1$. Finally, if $q' = 1$, we obtain $\alpha_1 = \zeta_q$ and

$$(\alpha_1, \zeta_q) = (\zeta_q, \zeta_q) = 1.$$

and if $q' \geq p$,

$$(\alpha_1, \zeta_q) = (\alpha_1, \alpha_1)^{q'} = 1$$

since $(\alpha_1, \alpha_1) = 1$ if $p \neq 2$, and $(\alpha_1, \alpha_1) = 1$ or $\zeta_q^{q/2}$ if $p = 2$.

THEOREM 2. [6: p. 104]. *Let* $n \equiv 1 \pmod 2$. *Then there exists a basis of* $L^\times$ mod $L^{\times 2}$ *satisfying the following properties*:

(I) π *is a uniformizing parameter of* L;

(II) α_0 *is 2-primary*;

(III) $\alpha_0, \alpha_1, \cdots, \alpha_n \in U_L$;

(IV) $(\alpha_0, \pi) = -1$, $(\alpha_1, \alpha_1) = -1$;

$(\alpha_{2\nu+1}, \alpha_{2\nu}) = (\alpha_{2\nu}, \alpha_{2\nu+1}) = -1$ *for* $\nu = 1, \cdots, (n-1)/2$, *and for all other pairs of basic elements the local symbol equals* 1.

(*V*) $\alpha_1 = -1$.

Let us remark that, in the situation of theorem 2, the extension $L(\sqrt{-1})/L$ is totally ramified (thus $q' = 1$), since $n \equiv 1 \pmod 2$ and $Q_2(\sqrt{-1})/Q_2$ is totally ramified (the prime 2 ramifies in $Q(\sqrt{-1})/Q$)).

PROPOSITION 4. *Let* L/Q_p, $q > 1$. *Then* $(\mu_q, L_a/L) \approx \mu_q$.

Proof. It suffices to show the existence of a $\beta \in L^\times$ mod $L^{\times q}$ such that $(\mu_q, L(\sqrt[q]{\beta})/L) \approx \mu_q$, since this, *a fortiori*, implies that $(\mu_q, L_a/L) \approx \mu_q$. But, for $n \equiv 1 \pmod 2$, $(-1, -1) = -1$, and for $n \equiv 1 \pmod 2$ we get

$$(\zeta_q, \alpha_2) = \zeta_q^{q'} = \zeta_q \quad \text{if} \quad q' = 1$$

and

$$(\zeta_q, \pi) = \zeta_q^{-(g-1)/q} \quad \text{if} \quad 1 < q' \leq q,$$

where $(g-1)q \not\equiv 0 \pmod p$, which proves the proposition.

Before proving our assertion about condition (P) being equivalent to a statement on the reciprocity, we need some easy lemmata:

LEMMA 1. *If* $L \subseteq M \subseteq M \subseteq L_q$, *then* $(\mu_q, M/L) = 1$.

Proof. Since $Gal(L_q/L) \approx (Z/qZ)' \times H$, where H is torsion-free, we have $(\mu_q, L_q/L) \subseteq (Z/qZ)'$, and, *a fortiori*, $(\mu_q, L_q/L) = 1$, since the elements of $(Z/qZ)'$ have orders prime to p. Whence $(\mu_q, M/L) = 1$ if $L \subseteq M \subseteq L_q$.

LEMMA 2. *Let* L/Q_p *as before, and* $1 < \hat{q} \leq q$, $\hat{q}$ *a power of* p. *Then* *a)* $\beta \notin L^{\times \hat{q}} \Rightarrow \beta \notin L^{\times q}$

b) If $L(\sqrt[\hat{q}]{\beta})/L$*, cyclic of degree* $\hat{q}$*, and* L_q/L *are linearly disjoint, then* $L(\sqrt[q]{\beta})/L$ *and* L_q/L *are linearly disjoint.*

We now prove

THEOREM 3. *Let* $L(\sqrt[\hat{q}]{\beta})/L$ *be an extension of degree* $\hat{q}$, $1 < \hat{q} \leq q$, $\hat{q}$ *a power of* p. *Let* $\tau(\sqrt[\hat{q}]{\beta}) = \zeta_{\hat{q}} \cdot \sqrt[\hat{q}]{\beta}$ *(so that* τ *generates* $Gal(L(\sqrt[\hat{q}]{\beta})/L)$. *Then the following statements are equivalent:*

a) $L(\sqrt[\hat{q}]{\beta})/L$ *and* L_q/L *are linearly disjoint.*

b) $(\zeta_q, L(\sqrt[\hat{q}]{\beta})/L) = \tau^a$, $a \not\equiv 0 \pmod p$.

Proof. a) $\Rightarrow$ *b)*. By lemma 2, $L(\sqrt[q]{\beta})/L$ and L_q/L are linearly disjoint, so $Gal(L_q(\sqrt[q]{\beta})/L) \approx Gal(L_q/L) \times Gal(L(\sqrt[q]{\beta})/L) \approx [\lim Gal(K/L)] \times Gal(L(\sqrt[q]{\beta})/L)$, where K runs over all intermediate fields $L \subseteq K \subseteq L_q$, K/L finite. Hence

$$(\zeta_q, L_q(\sqrt[q]{\beta})/L) = [\varprojlim(\zeta_q, K/L)] \times (\zeta_q, L(\sqrt[q]{\beta})/L) = (\zeta_q, L(\sqrt[q]{\beta})/L)$$

(lemma 1)

$$= \tau_1^{\,a}, \text{ where } a \not\equiv 0 \pmod p,$$

by proposition 2, where $\tau_1(\sqrt[q]{\beta}) = \zeta_q \sqrt[q]{\beta}$, and τ_1 generates $Gal(L(\sqrt[q]{\beta})/L)$. But then $(\zeta_q, L_q(\sqrt[\hat{q}]{\beta})/L) = (\zeta_q, L(\sqrt[\hat{q}]{\beta})/L) = \tau^a$, with $\tau = \tau_1 |_{L(\sqrt[\hat{q}]{\beta})}$, since $\tau(\sqrt[\hat{q}]{\beta}) = \zeta_{\hat{q}} \sqrt[\hat{q}]{\beta}$ (proposition 3).

b) $\Rightarrow$ *a)* By lemma 2, a), $\beta \notin L^{\times q}$. Also, since $a \not\equiv 0$ (mod p), we have $(\zeta_q, L(\sqrt[p]{\beta})/L) = \tau_1^{\,a} \neq 1$, where $\tau_0 = \tau |_{L(\sqrt[p]{\beta})}$. Now $L(\sqrt[p]{\beta}) \cap L_q = L$, since otherwise $L(\sqrt[p]{\beta}) \subset L_q$ would imply $(\zeta_q, L(\sqrt[p]{\beta})/L) = 1$, by lemma 1. But then, for any subextension M/L, $M \subseteq L_q$, we have

$$Gal(M(\sqrt[p]{\beta})/L) \approx Gal(M/L) \times Gal(L(\sqrt[p]{\beta})/L);$$

hence M/L and $L(\sqrt[p]{\beta})/L$ are linearly disjoint. Since M/L was arbitrary, the result follows from lemma 2, *b)* and proposition 6, chap. V, §2[3].

PROPOSITION 5. *The extensions* $L(\sqrt[\hat{q}]{\beta})/L$ *which are linearly disjoint from* L_q/L *are totally ramified.*

Proof: Since $Gal(L_{nr}/L) \approx \hat{Z}$, we have $\mu_q = Gal(L_a/L_q) \subseteq Gal(L_a/L_{nr})$, and therefore $L_q \subseteq L_{nr}$, by the Galois correspondence. Consequently, L_{nr}/L and $L(\sqrt[\hat{q}]{\beta})/L$ are linearly disjoint [3: prop. 6, chap. V, §2], whence $L(\sqrt[\hat{q}]{\beta})/L$ is totally ramified.

§3. Determination of the extensions $L(\sqrt[q]{\beta})/L$ satisfying property P

In this section we are going to use the results in the previous ones to determine all $\beta \in L^\times \bmod L^{\times q}$ for which condition P is satisfied. The following result in total ramification (especially condition c)) will play a crucial role.

THEOREM 4. *Let L/Q_p be a local field and q the highest power of p for which the q-th roots of 1 are contained in L. Then the following statements are equivalent*:

a) $L(\sqrt[q]{\zeta_q})/L$ *is totally ramified.*

b) $(-, L(\sqrt[q]{\zeta_q})/L)$: $U_L \to Gal(L(\sqrt[q]{\zeta_q})/L$ *is onto.*

c) *There is a uniformizing parameter* π *of* L *such that* $(\pi, \zeta_q) = 1$.

d) $X^p \equiv \zeta_q (mod\ \mathcal{P}_L{}^{pe/(p-1)}$, *has no solutions in* L, *where* $e = e(L/Q_p)$ *is the ramification index of* L/Q_p.

Remark. If $[L:Q_p] \equiv 1 \pmod 2$, this is always the case as remarked before.

Proof: The equivalence of *a*) and *b*) follows trivially from the fact that $(U_L, L(\sqrt[q]{\zeta_q})/L) = 1$, the inertia group of $L(\sqrt[q]{\zeta_q})/L$, whose order is precisely $e(L(\sqrt[q]{\zeta_q})/L)$ [4: p. 224]. That of *a*) and *b*) follows from Satz 119 in [5] for $p = q$, and the fact that $L(\sqrt[q]{\alpha})/L$ is totally ramified if, and only if, $L(\sqrt[p]{\alpha})/L$ is totally ramified. Let us show that *c*) $\Rightarrow$ *b*) and that *a*) $\Rightarrow$ *c*) to complete the proof.

$c \Rightarrow b$) We know that $(-, L(\sqrt[q]{\zeta_q})/L)$: $L^\times \to Gal(L(\sqrt[q]{\zeta_q})/L)$ is onto; hence, using the fact that any $\gamma \in L^\times$ can be written uniquely as $\pi^m \epsilon, m \in Z, \epsilon \in U_L$, and also that $(\pi, L(\sqrt[q]{\zeta_q})/L) = 1$, by hypothesis, we obtain $Gal(L(\sqrt[q]{\zeta_q})/L) = (L^\times, L(\sqrt[q]{\zeta_q})/L) = (U_L, L(\sqrt[q]{\zeta_q})/L)$; consequently, $(-, L(\sqrt[q]{\zeta_q})/L)$: $U_L \to Gal\ (L(\sqrt[q]{\zeta_q})/L)$ is onto.

a) $\Rightarrow$ *c*) Since $L(\sqrt[q]{\zeta_q})/L$ is totally ramified, we may take $g = q' = 1$ and $\zeta_q = \alpha_1$. Hence $(\pi, \zeta_q) = 1$ if $n \equiv 0 \pmod 2$ (corollary to theorem 1) and $(\pi, -1) = 1$ if $n \equiv 1 \pmod 2$ ((IV), theorem 2).

We are now ready to determine the extensions $L(\sqrt[q]{\beta})/L$, $\beta \in L^\times\ mod\ L^{\times q}$, satisfying condition *P*. This determination will depend on the way $L(\sqrt[q]{\zeta_q})/L$ ramifies, that is on theorem 4.

THEOREM 5. *If* $L(\sqrt[q]{\zeta_q})/L$ *is not totally ramified, then* $L(\sqrt[q]{\beta})/L$ *satisfies condition P if, and only if,* $L(\sqrt[q]{\beta}) = L(\sqrt[q]{\pi'})$, *where* π' *is a uniformizing parameter. Moreover, there are exactly* q^{n+1} *such extensions.*

Proof: ($\Rightarrow$) Suppose that $L(\sqrt[q]{\beta})/L$ satisfies condition *P*, and let

$$\beta = \pi^m \alpha_o^{a_0} \alpha_1^{b_1} \alpha_2^{b_2} \cdots \alpha_n^{b_n}\ mod\ L^{\times q}$$

in the basis described in theorem 1 (recall that necessarily $n \equiv 0 \pmod 2$). Then $m \not\equiv 0 \pmod q$, for otherwise we contradict *b*) in theorem 3, since

$$(\beta, \zeta_q) = \zeta_q^{-q' b_2} \text{ and } q' b_2 \equiv 0 \pmod p,$$

because $q' \geq p$ (same notations as in theorem 1). Furthermore,

$$(\beta, \zeta_q) = \zeta_q^{m(g-1)/q - q' b_2}$$

and

$$m(g-1)/q - q' b_2 \equiv m(g - 1/q) \not\equiv 0 \pmod p;$$

thus $m \not\equiv 0 \pmod p$ since $(g-1)/q \not\equiv 0 \pmod p$ (theorem 1). Taking $\pi' =$

$\beta^x(\pi^y)^q$, with $mx + qy = 1$, it follows that $L(\sqrt[q]{\pi'}) = L(\sqrt[q]{\beta})$, since x and q are relatively prime and clearly $\nu(\pi') = mx + qy = 1$ [5: p. 151].

($\Leftarrow$) Let $L(\sqrt[q]{\beta}) = L(\sqrt[q]{\pi'})$, π' being a uniformizing parameter of L, say

(4) $\pi' = \pi\alpha_0^{p_0}\alpha_1^{p_1}\alpha_2^{p_2} \cdots \alpha_n^{p_n}$ *mod* $L^{\times q}$.

Then

$$(\beta, \zeta_q) = (\pi', \zeta_q)^r = \zeta_q^{r[(g-1)/q - p_2q']}$$

where $\beta = \pi'^r\gamma^q, \gamma \in L^\times$, r and q being relatively prime. Since $(g-1)/q - p_2q' \not\equiv 0 \pmod p$, where $\beta = \pi'^r\gamma^q$, $\gamma \in L^\times$, r and q being relatively prime. Since $(g-1)/q - p_2q' \not\equiv 0 \pmod p$, it follows, by virtue of theorem 3, that $L(\sqrt[p]{\beta})/L$ satisfies P.

Finally, from (4) it follows that there are exactly q^{n+1} such extensions.

In order to elaborate in this case a complete list of such extensions, we may use the basis, if available; otherwise, starting from a given uniformizing parameter we should be able to exhaust all the possibilities in a finite number of steps.

We now turn to the remaining case: $L(\sqrt[q]{\zeta_q})/L$ is totally ramified.

THEOREM 6. *Let us suppose that* $L(\sqrt[q]{\zeta_q})/L$ *is totally ramified. Then*:

a) *There are exactly* $\varphi(q)q^{n+1}$ *distinct extensions* $L(\sqrt[q]{\beta})/L$, $\beta \in L^\times$ *mod* $L^{\times q}$ *satisfying property* P.

b) *If* $n \equiv 1$ (*mod* 2) *and* $\{\pi, \alpha_0, \alpha_r, \cdots, \alpha_n\}$ *is the basis in theorem* 2, *then* $L(\sqrt[q]{\beta})/L$, *with*

$$\beta = \pi^m\alpha_0^{b_0}\alpha_1^{b_1}\alpha_2^{b_2} \cdots \alpha_n^{b_n} \quad mod \quad L^{\times 2},$$

satisfies property P *if, and only if*, $b_1 \equiv 1$ (*mod* 2).

c) *If* $n \equiv 0$ (*mod* 2) *and* $\{\pi, \alpha_0, \alpha_1, \cdots, \alpha_n\}$ *is the basis in theorem* 1, *then* $L(\sqrt[q]{\beta})/L$, *with*

$$\beta = \pi^m\alpha_0^{b_0}\alpha_1^{b_1}\alpha_2^{b_2} \cdots \alpha_n^{b_n} \quad mod \quad L^{\times q},$$

satisfies property P *if, and only if*, $b_2 \not\equiv 0$ (*mod* p).

Proof: Since in both cases we may take $g = q' = 1$ and $\alpha_1 = \zeta_q$, we get

$$(\beta, \zeta_q) = (\beta, -1) = (-1, -1)^{b_1} = (-1)^{b_1} \quad \text{if} \quad n = (mod\ 2),$$

and

$$(\beta, \zeta_q) = (\alpha_2, \zeta_q)^{b_2} = \zeta_q^{-b_2} \quad \text{if} \quad n \equiv 0\ (mod\ 2).$$

a), *b*), *c*) follow now and easily.

In order to use theorem 6 to elaborate a list of the extensions satisfying P, in the case there considered, we shall need always an explicit basis of $L^\times$ *mod* $L^{\times q}$. Since $K = Q_p(\zeta_q) \subseteq L$ and we have $(\beta, \zeta_q) = (N_{L/K}(\beta), \zeta_q)$, for all $\beta \in L^\times$, we conclude that $L(\sqrt[q]{\beta})/L$ satisfies P if, and only if, $K(\sqrt[q]{\beta'})/K$, $\beta' = N_{L/K}(\beta)$, satisfies P, because $K(\sqrt[q]{\zeta_q})/K$ is always totally ramified [2: p. 277, exc. 4]. Explicit bases for the case contemplated in theorem 6 are computed in [1].

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