GENERALIZATIONS OF STONE AND SHIROTA THEOREMS TO PARTIALLY ORDERED TOPOLOGICAL SPACES

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Introduction and Notation

Let **PTop** be the category of all Partially Ordered Topological Spaces with continuous, isotone functions as their morphisms. Let **CrORR** be the subcategory of $\boldsymbol{R}$-regular spaces in **PTop**, **COTS** the subcategory of **PTop**-spaces with compact underlying topological space and continuous partial order, and $\boldsymbol{R}$**cOT** the subcategory of $\boldsymbol{R}$-compact spaces in **PTop** (see [1] pag. 97 and 100).

The primary object of this paper is to extend to $\boldsymbol{R}$**cOT** the theorem given in [10] pag. 127, by T Shirota, which for real compact topological spaces X states that the lattice CX determines the space. This result supercedes earlier results of Kaplansky [5] about the lattice CX; of M. H Stone [11] about the ring CX; and of A. N. Milgram [6] about the multiplicative semigroup CX, for X compact Hausdorff, and of T. Shirota [9] for the translation lattice and for the semigroup CX, and of E. Hewitt [4] for ring CX where X is real compact.

Given a topological space T, we can realize it as the partially ordered topological space (T, d) where d is the discrete partial order (no two elements are comparable).

In this sense **PTop** is an extension of **Top**, the category of all topological spaces.

We call C_1X the set of continuous isotone, real valued functions on a partially ordered topological space $X = (T, \leq)$. This set is a subset of $CT = C_1(T, d)$.

By abuse of the language if $X = (T, \leq)$, we write CX instead of CT. If our attempt to an straight-forward generalization had been successful, the small set C_1X would have provided the information not only on the topology of X, but on its partial order as well.

Leaving for a future work our initial aim, we restricted ourselves in this work to compact Hausdorff X and to rings, l-rings, l-groups and translation lattices generating them with C_1X when necessary. This led to counterexamples for rings, l-rings and for pointed l-groups. See Section 2, Theorem 2.

However, in Section 3, we do define categories of pairs with first component a ring, a l-ring, a pointed l-group, or a pointed translation lattice, and display new objects which actually characterize compact ordered topological spaces. In some of these cases (CX, C_1X) characterizes real-compact ordered topological spaces.

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In our Theorem 3, for example, we realize CX in a natural way as a pointed translation lattice, and C_1X as a subset of CX.

We introduce the category p-**PTL** of pairs (A, B) where A is a pointed translation lattice and $B \subset A$ with its morphisms defined in the obvious way.

Theorem 3 states that for $X, Y \in$ ***R*c****OT**, if $\psi:(CY, C_1Y) \cong (CX, C_1X)$ is a p-**PTL** isomorphism, implies the existence of $f:X \cong Y$ in ***R*c****OT** such that $\psi = C_1(f)$, where C_1 is considered as a natural ***R*c****OT** $\rightarrow$ **PTL** functor.

An application of this theorem for those spaces with discrete partial order yields that for X, Y real compact spaces, if $\psi:CY \cong CX$ is a **PTL**-isomorphism, there is $f:X \cong Y$ homeomorphism such that $\psi = C(f)$.

Theorems 4 and 5, are similar results this time considering the natural p-Ring-structure and p-pointed-ℓ-group-structure respectively.

A Corollary of Theorem 4 is Shirota Theorem, according to which if $\psi:CY = CX$ is a ring-isomorphism, there is $f:X \cong Y$ a homeomorphism such that $\psi = C(f)$.

Sharper results are given in Section 4 for $X \in$ **COTS**:

Let AX, LrX and LgX be respectively the subring, the sub-ℓ-ring and the sub-ℓ-group of CX generated by C_1X. Our Theorem 10 shows that for $X, Y \in$ **COTS**, if $\psi:(AY, C_1Y) \cong (AX, C_1X)$ is a p-ring isomorphism, there exists $\lambda:X \cong Y$ a **COTS**-isomorphism such that $\psi = C_1(\lambda)$, and similarly if $\psi:(LrY, C_1Y) \cong (LrX, C_1X)$ is a p-ℓ-ring isomorphism.

A Corollary of Theorem 10, Corollary 5, is the Stone Theorem, for X compact and CX a ring.

Finally these technics are adapted for pointed-ℓ-groups and the corresponding theorem (Theorem 11) is proved.

1. General statements about C_1X

Let F be a functor from a subcategory of **PTop** to a category of algebras. FX may characterize X, but we are interested in whether the characterization happens in such a way that $\varphi:FX \cong FY$, implies the existence of $f:Y \cong X$ such that $\varphi = F(f)$ and $F(f)(h) = h \circ f$.

LEMMA *1. Every space S in* **CrORR** *has the initial* **PTop**-*structure with respect to C_1X.*

Proof: By [1] Corollary 1 and Theorem 5, the evaluation map $\rho:X \rightarrow R^{C_1X}$ is an embedding. This shows that C_1X separates points of X, and by [1], there exists a **PTop**-initial structure X^i on X with respect to C_1X. Now consider $\rho':X^i \rightarrow R^{C_1X}$ given by $\rho'(x) = \rho(x)$ since $C_1X = C_1X^i$, then ρ' is also an embedding and we have $X \cong \rho X = \rho' X^i \cong X^i$.

THEOREM *1: Let* **A** *be a category such that C_1:***CrORR** $\rightarrow$ **A**, *$C_1(X) = C_1X$ and $C_1(f)(g) = gf$ defines a functor. Let $\varphi:C_1X \cong C_1Y$ in* **A**. *Then the following statements are equivalent*:

1) *There exists a bijection $f:X \rightarrow Y$ such that $p_x \circ \varphi = p_{f(x)}$ for all $x \epsilon X$*
2) *There exists a* **CrORR**-isomorphism *$f:X \rightarrow Y$ such that $C_1(f) = \varphi$.*

Proof: Let $f:X \to Y$ be a bijection as in 1). Let $h \epsilon C_1Y$, and $x \epsilon X$ be arbitrary. Then $\varphi(h)(x) = (p_x \circ \varphi)(h) = p_{f(x)}(h) = (h \circ f)(x)$.

Therefore $\varphi(h) = h \circ f$. To prove 2) it is then sufficient to show that f is continuous and isotone, since the same argument will give f^{-1} continuous and isotone. Since φ is bijective, we have $C_1X = \{h \circ f \mid h \epsilon C_1Y\}$. Since Y has initial structure with respect to C_1Y, it follows that f is continuous and isotone. Conversely, suppose $f:X \to Y$ is an isomorphism in **CrORR**, such that $C_1(f) = \varphi$. Clearly f is bijective and since $(p_x \circ \varphi)(h) = \varphi(h)(x) = C_1(f)(h)(x) = (h \circ f)(x) = h(f(x)) = p_{f(x)}(h)$, we obtain 1).

Remark: If we define a partial order on C_1X by $f \leqq g$ if and only if $f(x) \leqq g(x)$ for all $x \epsilon X$, then $C_1(X)$ is a lattice.

The following statement is false: "*If* $\varphi:C_1Y \to C_1X$ *is a lattice isomorphism, there exists an isomorphism in* **COTS**, $f:X \to Y$ *such that* $\varphi = C_1(f)$"

Proof: Let $X = Y = \mathbf{2}$ where $\mathbf{2} = (\{0.1\}, 0 \leq 1)$ with discrete topology and denote by (a, b) the function $(a, b):X \to \boldsymbol{R}$ where $(a, b)(0) = a$ and $(a, b)(1) = b$. Define $\psi:C_1Y \to C_1X$ by $\psi(a, b) = (2a + 1, 2b + 1)$. Obviously ψ is a lattice isomorphism, but the above statement would imply the existence of a bijective $f:X \to Y$ such that $p_0 = \psi = p_{f(0)}$ which means:

$$3 = p_0(3, 3) = (p_0 \psi)(1, 1) = p_{f(0)}(1, 1) = 1$$

a contradiction.

Remark: The above example leaves open the question of whether there exists $f:X \cong Y$ in **COTS**, such that $C_1(f)$ is another isomorphism of C_1X and C_1Y, but we include this example here because the statements which we shall prove later are of the type just discussed.

2. Counterexamples

Since C_1X fails in general to have the algebraic structures considered for CX we could try to generalize the theorems concerning CX by considering the subalgebras of CX generated by C_1X.

Let F: **CrORR** $\to$ **Cr** be the order-forgetful functor. U:**Cr** $\to$ **CrORR** the canonical inclusion, given by $X \to (X, d)$. We set $CX = \mathbf{Cr}(FX, \boldsymbol{R})$, AX for the subring of CX generated by C_1X, LrX for the sub-l-ring of CX generated by C_1X and LgX for the sub-l-group of CX generated by C_1X. As an example we remark that $A([a, b])$ is the set of continuous functions on $[a, b]$ which are of bounded variation.

We shall introduce more functors when we will need them.

THEOREM 2: *The following statements for* $X, Y \epsilon$ **COTS** *are false:*
1) $AY \cong AX$ *in the category of rings, then* $X \cong Y$ *in* **COTS**
2) $LrY \cong LrX$ *in the category of l-rings, then* $X \cong Y$ *in* **COTS**.
3) $LgY \cong LgX$ *in the category of l-groups, then* $X \cong Y$ *in* **COTS**.

Proof: If we describe as in a previous remark, the functions in $C_1\mathbf{2}$ by $(a, b):\mathbf{2} \to \boldsymbol{R}$ such that $(a, b)(0) = a$ and $(a, b)(1) = b$, $C_1\mathbf{2}$ is $\{(a, b) \in \boldsymbol{R}^2 \mid a \leqq b\}$ while $C\mathbf{2}$ is the whole $\boldsymbol{R}^2$. Since $a \nleqq b$ implies $b \leqq a$ and therefore $-a \leqq -b$, the group generated by $C_1\mathbf{2}$ is $C\mathbf{2}$. Let $\mathit{2} = (\{0, 1\}, d)$ where d is the discrete partial order. Now is $C\mathit{2} = C\mathbf{2}$ and so is $Lg\mathbf{2} = Lr\mathbf{2} = A\mathbf{2} = C\mathbf{2} = C\mathit{2} = A\mathit{2} = Lr\mathit{2} = Lg\mathit{2}$. But it is clear that $\mathbf{2} \ncong \mathit{2}$.

3. Generalization of theorems on CX

We introduce categories of pairs, with first component a certain algebraic system and second component a subset of the underlying set of the first. We show for **RcOT** and for some of these algebraic structures that the pair (CX, C_1X) characterizes the space X. We leave for our Section 4 characterizations which depend more strongly on C_1X.

Notation: If **AK** denotes a category of algebras, we denote by **p-AK** the category whose objects are pairs (X, Y) such that $X \in$ **AK** and $Y \subset X$ and whose morphisms are $m:(X, Y) \to (Z, W)$ where $m:X \to Z$ is a **AK**-homomorphism and $m(Y) \subset W$.

We include the following definition for the convenience of the reader:

Definition 1. (Shirota [9]): By a *translation lattice* L we mean a lattice where for every $a \in L$ and for real numbers α, a sum $a + \alpha$ is defined which satisfies the following conditions:

1) $a + 0 = a$
2) $(a + \alpha) + \beta = a + (\alpha + \beta)$
3) If $\alpha \geqslant 0$, then $a + \alpha \geqslant a$
4) If $a \geqslant b$, then $a + \alpha \geqslant b + \alpha$.

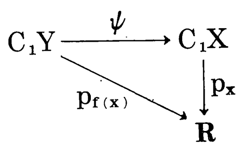
Remark: If L is a translation lattice, every real number r induces on L an unary operation $\bar{r}:L \to L$, given by $\bar{r}(a) = a + r$. $C(X, \boldsymbol{R})$ can obviously be considered a translation lattice by setting $(f + \alpha)(x) = f(x) + \alpha$ for a real number α and for a function $f \in CX$.

Definition 2. A translation lattice L with a nullary operation $z \in L$ will be called a pointed translation lattice. We shall denote by **PTL** the corresponding category.

Remark: Clearly, CX and C_1X are pointed translation lattices, where we shall choose as its point the constant zero function. A **PTL** homomorphism will be of course a function $f:L \to L'$ such that $f(z) = z'$, $f(a \wedge b) = f(a) \wedge f(b)$ and $f(a + r) = f(a) + r$.

THEOREM 3: *If $X, Y \in$ **RcOT** and $\psi: (CY, C_1Y) \to (CX, C_1X)$ is a* **p-PTL**-*isomorphism, there exists $f:X \to Y$ a* **RcOT**-*isomorphism such that $\psi = C_1(f)$.*

Proof: For every $x \in X$, the map $p_x:CX \to \boldsymbol{R}$ given by $p_x(f) = f(x)$ is a translation lattice-homomorphism. If we consider $CY \xrightarrow{\psi} CX \xrightarrow{p_x} \boldsymbol{R}$ in the category **TL** (translation lattices), it follows by [9] Theorem 8, pag. 35, that there exists a unique point which we call $f(x)$ such that $p_x \cdot \psi = p_{f(x)}$; the uniqueness arising from the fact that ρ_Y is injective. We show that the associating rule $x \to f(x)$ defines a bijective function. By the uniqueness of $f(x)$, it is a function. Let $f(x) = f(y)$; then $p_x \cdot \psi = p_{f(x)} = p_{f(y)} = p_y \cdot \psi$ and since ψ is an isomorphism and hence surjective, $p_x = p_y$. This shows for $X \in \boldsymbol{R}$c**OT** that $x =$ y. To show that f is surjective, let $y \in Y$ and consider $CX \xrightarrow{\psi^{-1}} CY \xrightarrow{p_y} \boldsymbol{R}$. By the same argument as above, there exists a unique element of X, $g(y)$ such that $p_y \cdot \psi^{-i} = p_{g(y)}$. Therefore: $p_y = p_y \cdot \psi^{-i} \cdot \psi = p_{g(y)} \cdot \psi$ which means that $y = f(g(y))$. We have that $\psi(C_1X) \cong C_1Y$, that f is bijective and that, for every $x \in X$, the following diagram commutes:



It then follows from Theorem 1 that $f\colon X \to Y$ is a $\boldsymbol{R}$c**OT**-isomorphism, and since

$$\psi(h)(x) = p_x \cdot \psi(h) = p_{f(x)}(h) = h(f(x)) = (h \cdot f)(x),$$

then

$$\psi(h) = h \cdot f = C_1(f)(h).$$

COROLLARY *1*: *If X, Y are real compact spaces, and $\psi\colon CY \to CX$ a* **PTL**-*isomorphism, there exists a homeomorphism $f\colon X \to Y$ such that $\psi = C(f)$.*

Proof: We simply note that $C_1X = CX$ and $C_1Y = CY$.

LEMMA *2*: *For every completely regular topological space X, let SAX be a subring of CX which contains all the constant functions. If $h : SAY \to SAX$ is a surjective ring homomorphism, then $h(\bar{r}) = \bar{r}$ for all $r \in \boldsymbol{R}$.*

Proof: For $r = 1$, since $\bar{1} \in SAX$ and h is surjective, there exists $g \in SAY$ such that

$$\bar{1} = h(g) = h(g \cdot \bar{1}) = h(g) \cdot h(\bar{1}) = \bar{1} \cdot h(\bar{1}) = h(\bar{1}).$$

Suppose $h(\bar{n}) = \bar{n}$ for a positive integer n. Then

$$h(\overline{n+1}) = h(\bar{n} + \bar{1}) = h(\bar{n}) + h(\bar{1}) = \bar{n} + \bar{1} - \overline{n+1}$$

One shows easily that $h(\bar{0}) = \bar{0}$ and if n is a negative integer

$$\bar{0} = h(\overline{-n+n}) = h(\overline{-n} + \bar{n}) = h(\overline{-n}) + h(\bar{n}) = \overline{-n} + h(\bar{n}).$$

Therefore $h(\bar{n}) = \bar{n}$. Moreover denoting by r the rational $1/m$, one easily obtains

$\bar{r} = \bar{1}/\bar{m}$ and $h(\bar{r}) = \bar{r}$, and from this follows just as easily $h(\bar{r}) = \bar{r}$ for all rationals r. Finally if $r \in \boldsymbol{R}$ and $r > 0$, there exists $s \in \boldsymbol{R}$, $r = s^2$. Then $h(r) = h(s) \cdot h(s) > 0$. Let $r \in \boldsymbol{R}$ and $r = \lim_{n \in \boldsymbol{N}} r_n$ where r_n is rational for all $n \in \boldsymbol{N}$. Let ϵ be an arbitrary positive rational number. Let $N \in \boldsymbol{N}$ be such that whenever $n > N$, $r_n - r < \epsilon$ or $r - r_n < \epsilon$ and let $n > N$. If $r_n - r < \epsilon$, $\epsilon + r - r_n > 0$, and $\bar{\epsilon} + h(\bar{r}) - h(\bar{r}_n) = h(\overline{\epsilon + r - r_n}) > 0$. Therefore $\bar{r}_n - h(\bar{r}) < \bar{\epsilon}$ and similarly if $r - r_n < \epsilon$, then $h(\bar{r}) - \bar{r}_n < \bar{\epsilon}$. For every $x \in X$, we then have: $r_n - h(\bar{r})(x) < \epsilon$ or $h(\bar{r})(x) - r_n < \epsilon$ which means that $h(\bar{r})(x) = \lim_{n \in \boldsymbol{N}} r_n = r$ for all $x \in X$, and can be expressed as $h(\bar{r}) = \bar{r}$.

THEOREM *4*: *If* X, $Y \in$ **RcOT**, *and* $\psi: (CY, C_1Y) \to (CX, C_1X)$ *is a* **p-Ring**-isomorphism there exists an **RcOT**-isomorphism $f: X \to Y$ *such that* $\psi = C_1(f)$.

Proof: Since $\psi: CY \to CX$ is a **Ring**-isomorphism, it can be interpreted as **PTL**-isomorphism because $\psi(0) = 0$ and $\psi(f + \bar{r}) = \psi(f) + \psi(\bar{r}) = \psi(f) + \bar{r}$. By Theorem 3 we obtain the desired result.

COROLLARY *2*: *If* X, Y *are real-compact spaces, and* $\psi: CY \to CX$ *a ring isomorphism, there exists an homeomorphism* $f: X \to Y$ *such that* $\psi = C(f)$

Proof: The proof is as in Corollary 1.

Definition 3: An ℓ-group G with a nullary operation $1 \in G$ will be called a pointed ℓ-group. A **PLG**-homomorphism $h: G \to G'$ is an ℓ-group homomorphism such that $h(1) = 1'$.

THEOREM *5*: *If* X, $Y \in$ **RcOT**, *and* $\psi: (CY, C_1Y) \to (CX, C_1X)$ *is a* **p-PLG**-isomorphism there exists an **RcOT**-isomorphism $f: X \to Y$, *such that* $\psi = C_1(f)$.

Proof: As in Theorem 3, we consider for every $x \in X$, $CY \xrightarrow{\psi} CX \xrightarrow{p_x} \boldsymbol{R}$ in **PLG** (for CZ we select the nullary operation $\bar{1} \in CZ$ and for $\boldsymbol{R}$, $1 \in \boldsymbol{R}$). By [9] Theorem 10 pag. 36, considering its proof, there exists a point $f(x)$ in Y which is unique as $Y \in$ **RcOT**, such that $p_x \cdot \psi = p_{f(x)}$. By following now all the steps of the proof of Theorem 3, we finish this proof.

COROLLARY *3*: *if* X, Y *are real-compact spaces and* $\psi: CY \to CX$ *is a* **PLG**-*isomorphism, there exists an homeomorphism* $f: X \to Y$ *such that* $\psi = C(f)$.

Proof: The proof is the same as in Corollary 1.

4. Sharper algebraic characterizations of COTS-spaces

As in Section 2, let F: **CrORR** $\to$ **Cr** be the order-forgetful functor. Let $CX =$ **Cr**$(FX, \boldsymbol{R})$. Let AX, LrX and LgX be respectively the subring, the sub-ℓ-ring and the sub-ℓ-group of CX generated by C_1X.

We introduce the concept of a distinguished ideal of AX, and relate this with filters in a subset ZX of PX ordered by the set inclusion. By associating each element of X to the unique ultrafilter of ZX which converges to x, and those with maximal distinguished ideals, we are able to characterize X in **COTS** by

(AX, C_1X). In a similar way we do the same with (LrX, C_1X) and (LgX, C_1X) also. See Theorems 10 and 11.

Notation: We define $Z:AX \to \mathrm{P}X$ and $Z_1:LrX \to \mathrm{P}X$ by $Zf = f^{-1}(0)$ and $Z_1f = f^{-1}(0)$ respectively. We denote ZAX by ZX and Z_1LrX by Z_1X. By a filter in ZX we mean a filter in the set ZX ordered by the inclusion.

Remark:

i) $Z\bar{0} = X$ iv) $Zf\cdot g = Zf \cup Zg$
ii) $Z\bar{1} = \varnothing$ v) $Z(f^2 + g^2) = Zf \cap Zg = Z_1(|f| + |g|)$
iii) $Zf = Zf^n$ vi) if $g = f \wedge \bar{1}$, then $Z_1f = Z_1g$.

Definition 4: Let I be an ideal of the ring AX. We call I a *distinguished ideal* if $I \cap \{f \in AX \mid Zf = \varnothing\} = \varnothing$.

LEMMA 3: *The intersection of any family of distinguished ideals is itself a distinguished ideal.*

Proof: trivial.

THEOREM 6: *For every distinguished ideal I of AX, ZI is a filter of ZX conversely if ψ is a filter in ZX, $Z^{-1}\psi = \{f \in AX \mid Zf \in \psi\}$ is a distinguished ideal of AX.*

Proof: Since I is a distinguished ideal of AX, $\varnothing \notin ZI$. Let Zf, Zg be two sets in ZI such that $f, g \in I$. Then $f^2 + g^2 \in I$ and therefore $Zg \cap Zf = Z(f^2 + g^2) \in ZI$. Let $f \in I$ and $Zf \subset Zg \in ZX$. Then $gf \in I$ and $Zg = Zg \cup Zf = Z(g\ f) \in ZI$. Conversely, using the notation of section 2, we remark that $C_1UFX = CX$ and therefor $AUFX = CX$. Since ψ has the finite intersection property and $\psi \subset CZUFX$, it generates a filter $\bar{\psi}$ in $ZUFX$. By [3] Theorem 2.3 b) pag. 25, $Z^{-1}\bar{\psi}$ is a proper ideal of CX, i.e. $Zf \neq \varnothing$ whenever $f \in Z^{-1}\bar{\psi}$. Therefore $Z^{-1}\psi = Z^{-1}\bar{\psi} \cap AX$ is a distinguished ideal of AX.

Remark: The ideal generated by f and g is distinguished exactly if $Zf \cap Zg \neq \varnothing$. i. e. if $f^2 + g^2$ is non invertible.

LEMMA 4: *Let ψ be a filter in ZX, and I a distinguished ideal of AX. Then $ZZ^{-1}\psi = \psi$ and $I \subset Z^{-1}ZI$.*

Proof: This is trivial as we have defined $Z:AX \to \mathrm{P}X$ as a map.

COROLLARY 4: *If I is a maximal distinguished ideal of AX, then $I = Z^{-1}ZI$. If ψ is a filter in ZX, there exists the distinguished ideal $I = Z^{-1}\psi$ such that $\psi = ZI$.*

THEOREM 7: *For every distinguished ideal M of AX, $Z^{-1}ZM$ is a maximal distinguished ideal if and only if ZM is an ultrafilter.*

Proof: Let M be a distinguished ideal. Suppose $Z^{-1}ZM$ is a maximal distinguished ideal and let ψ be a filter in ZX such that $ZM \subset \psi$. We obtain that $Z^{-1}ZM \subset Z^{-1}\psi$. Since $Z^{-1}ZM$ is a maximal distinguished ideal, we have $Z^{-1}ZM = Z^{-1}\psi$. By Theorem 6 and Lemma 4 $ZM = ZZ^{-1}ZM = ZZ^{-1}\psi = \psi$.

Conversely suppose that ZM is un ultrafilter, and I a distinguished ideal such that $Z^{-1}ZM \subset I$. Then $ZM = ZZ^{-1}ZM \subset ZI$ and since ZM is an ultrafilter, $ZM = ZI$. Therefore $Z^{-1}ZM = Z^{-1}ZI$. This, together with $Z^{-1}ZM \subset I \subset Z^{-1}ZI$ gives $Z^{-1}ZM = I$ which shows that $Z^{-1}ZM$ is a maximal distinguished ideal.

LEMMA 5: *If u is an ultrafilter in ZX and $Zf \cap Zg \neq \emptyset$ for all $Zg \in u$, then $Zf \in u$. If M is a maximal distinguished ideal of AX and $Zf \cap Zg \neq \emptyset$ for all $g \in M$, then $f \in M$.*

Proof: This is immediate by the maximality of u and the fact that ZM is an ultrafilter.

LEMMA 6: *For $x \in X$, if $p_x: AX \to \boldsymbol{R}$ is the x-th projection, $Zp_x^{-1}(0)$ is an ultrafilter.*

Proof: $p_x^{-1}(0)$ is clearly a distinguished ideal since $x \in Zf$ for all $f \in p_x^{-1}(0)$. Since the map p_x is a surjective ring homomorphism, $AX/p_x^{-1}(0) \cong \boldsymbol{R}$ and $p_x^{-1}(0)$ is a maximal ideal. By Corollary 4, $p_x^{-1}(0) = Z^{-1}Zp_x^{-1}(0)$ and by Theorem 7, $Zp_x^{-1}(0)$ is an ultrafilter.

Notation: We denote $Zp_x^{-1}(0)$ by A_x.

Remark: In order to characterize all the maximal distinguished ideals of AX as sets of the form $p_x^{-1}(0)$, we need to discuss convergence in our filters.

Definition 5: If we denote by $N(x)$ the set of all (not necessarily open) neighborhoods of x, a filter ψ of ZX is said to *converge to* x if the set of all Zf in $N(x)$ belongs to ψ. This means that ψ converges to x if and only if $A_x \cap N(x) \subset \psi$. A point x is said to be an *adherence point of* ψ if $x \in \bigcap \psi$. If x is an adherence point of ψ, $Zg \cap Zf \neq \emptyset$ whenever $Zg \in N(x)$ and $Zf \in \psi$. Therefore, in this last case, there exists an ultrafilter u such that $\psi \subset u$ and u converges to x.

Definition 6: (Nachbin [7]). Let $(X, \leq)$ be a partially ordered set and $S \subset X$. We call S *decreasing* if, whenever $a \leq b$ and $b \in S$, $a \in S$. Similarly S will be *increasing* if from $a \leq b$ and $a \in S$, $b \in S$ follows.

We denote by LS the smallest decreasing subset of X containing S and by MS the corresponding smallest increasing set. Since decreasing and increasing have the intersection property, and X is both decreasing and increasing. LS and MS always exist.

Definition 7: (Nachbin [7]. $(X, \leq) \notin$ **PTop** will be said to be *normally ordered* if for every two disjoint closed subsets A, B of X such that A is decreasing and B increasing, there exist two disjoint open sets U, V such that U contains A and is decreasing, and V contains B and is increasing.

Notation: We denote by **NOR** the category of normally ordered spaces in **PTop** and by **NORC** the intersection of **NOR** with **HOTS**.

LEMMA 7: *If X is a locally compact space in* **NORC**, *every filter ψ in ZX converges to at most one point.*

Proof: Suppose ψ converges to two distinct points x and y. Without loss of generality $x \nleq y$. Since $\leq$ is continuous, there exist neighborhoods U of x and V of y such that $LV \cap MU = \emptyset$. Without loss of generality, since X is locally compact, U and V are compact and therefore LV, MU are two disjoint closed sets which satisfy the hypothesis of [7] Prop. 4 pag. 44. Let $f \in C_1X$ be such that $fLV = 0$ and $fMU = 1$. Then $LV \subset f^{-1}(0) = Zf$ and therefore $Zf \in N(y)$. Similarly $MU \subset (f-1)^{-1}(0) = Z(f-1)$ which means that $Z(f-1) \in N(x)$. Having assumed that ψ converges to x and to y, Zf, $Z(f-1) \in \psi$ which is a contradiction as $Zf \cap Z(f-1) = \emptyset$ and ψ is a filter.

LEMMA 8: *Let $X \in$* **COTS.** *If a filter ψ of ZX converges to x, $\bigcap \psi = \{x\}$.*

Proof: Since X is compact and ψ is a family of closed sets which satisfy the finite intersection property, $\bigcap \psi \neq \emptyset$. Let $y \neq x$. If $x \nleq y$ we reason as in Lemma 7 and find $f \in C_1X$ such that $f(y) = 0$ and $Z(f-1) \in N(x) \subset \psi$. Therefore $y \notin Z(f-1)$ and accordingly $y \notin \bigcap \psi$. If $y \nleq x$, we similarly find compact neighborhoods V of y and U of x such that $LU \cap MV = \emptyset$, and a function g in C_1X such that $gLU = 0$ and $gMV = 1$. It follows that $g^{-1}(0) = Zg \in N(x) \subset \psi$ and $y \notin Zg$.

LEMMA 9: *If an ultrafilter ψ of ZX has an adherence x, ψ converges to x.*

Proof: We show that $A_x \cap N(x) \subset \psi$. Let $Zf \in A_x \cap N(x)$. For every $Zg \in \psi$, $x \in Zg \cap Zf$. Since ψ is an ultrafilter, $Zf \in \psi$.

THEOREM 8: *Let $X \in$* **COTS.** *The ultrafilters in ZX are exactly $(A_x)_{x \in X}$.*

Proof: Since X is compact, by the finite intersection property every ultrafilter has an adherence point and therefore converges. We know from Lemma 6 that every A_x is an ultrafilter in ZX, and it is obvious that $A_x = Zp_x^{-1}(0)$ converges to x. Let ψ be an arbitrary ultrafilter in ZX which converges to x. By Lemma 8, $\bigcap \psi = \{x\}$. Therefore, every set Zf of ψ satisfies the follwing: $x \in Zf, f(x) = 0$, $f \in p_x^{-1}(0)$, $Zf \in Zp_x^{-1}(0) = A_x$. This means that $\psi \subset A_x$ and since ψ is an ultrafilter, $\psi = A_x$.

Remark: A_x is the unique ultrafilter of ZX which converges to x, and every filter ψ which converges to x is a subset of A_x.

THEOREM 9: *Let $X \in$* **COTS.** *The maximal distinguished ideals of AX are exactly $(p_x^{-1}(0))_{x \in X}$.*

Proof: By Lemma 6 $p_x^{-1}(0)$ is a maximal distinguished ideal. Conversely, let M be an arbitrary maximal distinguished ideal of AX. By Corollary 4, $M = Z^{-1}ZM$ and by Theorem 7, ZM is an ultrafilter. Since ZM satisfies the finite intersection property and X is compact, there exists an $x \in X$ such that

ZM converges to x. Therefore $ZM = A_x$ and $M = Z^{-1}ZM = Z^{-1}A_x = Z^{-1}Zp_x^{-1}(0) = p_x^{-1}(0)$.

Notation: We denote by MAX the set of all maximal distinguished ideals in AX, which by the above theorem is $\{p_x^{-1}(0) | x \in X\}$.

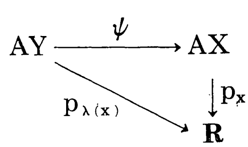
LEMMA *10*: *For every* $X \in$ **COTS**, *the natural map* $b_X: X \to MAX$ *given by* $b_X(a) = p_a^{-1}(0)$ *is a bijection.*

Proof: The surjectivity is obvious. Let $x \nleq y$ be given in X. Then $Mx \cap Ly = \varnothing$ and we can find a function $f \in C_1X \subset AX$ such that $f(y) = 0 \neq 1 = f(x)$. This means that $f \in p_y^{-1}(0)$ and $f \notin p_x^{-1}(0)$. Therefore $p_y^{-1}(0) \neq p_x^{-1}(0)$.

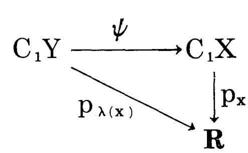
THEOREM *10*: *If* $X, Y \in$ **COTS**, *and* $\psi: (AY, C_1Y) \to (AX, C_1X)$ *is a* **p-Ring**-*isomorphism, there exists* $\lambda: X \to Y$ *a* **COTS**-*isomorphism such that* $\psi = C_1(\lambda)$. *Similarly if* $\sigma: (LrY, C_1Y) \to (LrX, C_1X)$ *is a* **p-ℓ-Ring**-*isomorphism there exists* $g: X \to Y$ *a* **COTS**-*isomorphism such that* $\sigma = C_1(g)$.

Proof: Let X, Y be spaces in **COTS** such that $\psi: (AY, C_1X) \to (AX, C_1X)$ is a **p-Ring**-isomorphism. Since $\psi: AY \to AX$ is a **Ring**-isopmorphism, it induces a bijective function $\bar{\psi}: MAY \to MAX$. Define λ as $b_Y^{-1} \circ \bar{\psi}^{-1} \circ b_X$. Let $x \in X$; then $p_{\lambda(x)}^{-1}(0) = b_Y(\lambda(x)) = (b_Y \circ \lambda)(x) = (\bar{\psi}^{-1} \circ b_x)(x) = \psi^{-1} p_x^{-1}(0)$.

Let $g \in p_{\lambda(x)}^{-1}(0)$; then $\psi(g)(x) = g(\lambda(x))$. We show next that $\psi(g) = g \circ \lambda$ for all $g \in AY$. Let $x \in X$ and $g \in AY$. Call $r = g(\lambda(x))$. Then $(g - \bar{r})(\lambda(x)) = 0$ and therefore $g - \bar{r} \in p_{\lambda(x)}^{-1}(0)$. But, for this case, we have just shown that $\psi(g - \bar{r})(x) = 0 = (g - \bar{r})(\lambda(x))$. By Lemma 2 $\psi(\bar{r}) = \bar{r}$ and we obtain $\psi(g)(x) - r = \psi(g)(x) - \psi(\bar{r})(x) = g(\lambda(x)) - \bar{r}(\lambda(x)) = (g \circ \lambda)(x) - r$. Since x was arbitrary, $\psi(g) = g \circ \lambda$, and we have shown that, for the bijection λ the following diagram commutes:



Since $\psi(C_1Y) = C_1X$ we can interpret this diagram as



By Theorem 1, it follows that $\lambda: X \to Y$ is a **CrORR**-isomorphism; i.e. a **COTS**-isomorphism.

The statement about (LrY, C_1Y) is proved in an analogous way.

COROLLARY *5*: *If X, Y are compact spaces and $\psi: CY \to CX$ is a* **ring**-*isomorphism, there exists an homeomorphism $\lambda: X \to Y$ such that $\psi = C(\lambda)$.*

Proof: The proof is the same as in Corollary 1.

Remark: We introduced in Definition 3 the concept of pointed ℓ-group (ℓ-group with a unit), **PLG**. If X is a **PLG** we call a subset Y a sub-ℓ-group with unit of X if Y is an ℓ-group and has the same unit as X. The intersection of a family of sub-ℓ-groups with unit is clearly a sub-ℓ-group with unit.

Notation: Let $X \in$ **COTS**. We denote by $Lgu(X)$ the sub-ℓ-group with 1 of CX which is the intersection of all L, sub-ℓ-groups with $\bar{1}$ of CX, which contain C_1X and such that, with $f \in L$ and f invertible in CX, $f^{-1} \in L$. We define $Z_2: Lgu(X) \to \mathrm{P}X$ by $Z_2 f = f^{-1}(0)$ and, as we did for Z and Z_1, we mean by Z_2X, $Z_2(Lgu(X))$.

LEMMA *11*: *Let $X \in$* **COTS**, *and $h: Lgu(X) \to \boldsymbol{R}$ be a surjective* **PLG**-*homomorphism. Then $h(\bar{r}) = \bar{r}$ for all $r \in \boldsymbol{R}$. If $h(f) = 0$, then $h(|f|) = 0$ and $Z_2 f \neq \varnothing$.*

Proof: By the definition of h, $h(\bar{1}) = 1$, and since $Lgu(X)$ is a lattice we can repeat the rest of the proof of Lemma 2, proceeding directly from $r_n - r < \epsilon$ to $r_n - h(\bar{r}) = h(\overline{r_n - r}) \leq h(\bar{\epsilon}) = \epsilon$.

If $h(f) = (0, h)(-f) = -h(f) = 0$. Therefore

$$h(|f|) = h(f \vee -f) = h(f) \vee h(-f) = 0 \vee 0 = 0.$$

Suppose $Z_2 f = \varnothing$. Then $0 \notin Imf$ and $f^{-1} \in CX$. Therefore $f^{-1} \in Lgu(X)$. We show that $h(|f^{-1}|) \neq 0$. Suppose $h(|f^{-1}|) = 0$. Since $h(|f|) = 0$, $h(|f| \, v \, |f^{-1}|) = 0$. But $\bar{1} \leq |f| \vee |f^{-1}|$ and $1 = h(\bar{1}) \leq h(|f| \vee |f^{-1}|) = 0$, a contradiction. Let $h(|f^{-1}|) = a > 0$, then $h(|f^{-1}| - \bar{a}) = h(|f^{-1}|) - h(\bar{a}) = a - a = 0$. It follows that $h((|f^{-1}| - \bar{a}) \vee |f|) = 0$. To obtain the contradiction needed to reject "$Z_2 f = \varnothing$", let $e > 2a$ and $d = \min\{a, e^{-1}\}$. We shall show that $(|f^{-1}| - \bar{a}) \vee |f \geqslant | \bar{d} > 0$. Since for each $x \in X$, $f(x) \neq 0$, it follows that $f^{-1}(x) \neq 0$ and $|f^{-1}(x)| \neq 0$. If $0 < |f^{-1}(x)| < 2a$, $e^{-1} < (2a)^{-1} < |f^{-1}(x)|^{-1} = |f(x)|$ and we obtain $d \leqslant e^{-1} < |f(x)|$. If $2a \leqslant |f^{-1}(x)|$, clearly $|f^{-1}(x)| - a \geqslant a \geqslant d$.

LEMMA *12*: *Let $X \in$* **COTS** *and $h: Lgu(X) \to \boldsymbol{R}$ be a surjective* **PLG**-*homomorphism. Then $Z_2 h^{-1}(0)$ is an untrafilter in Z_2X.*

Proof: Since h is a group-homomorphism, $h(\bar{0}) = 0$, and we have $\bar{0} \in h^{-1}(0)$, $Z_2\bar{0} = X \in Z_2 h^{-1}(0)$. Therefore $Z_2 h^{-1}(0) \neq \varnothing$. By Lemma 11 $\varnothing \in Z_2 h^{-1}(0)$. Let $f, g \in h^{-1}(0)$; then $|f| \vee |g| \in h^{-1}(0)$ and

$$Z_2 f \cap Z_2 g = Z_2(|f| \vee |g|) \in Z_2 h^{-1}(0).$$

To see that $Z_2 h^{-1}(0)$ is a filter in Z_2X, we need now show that if $f \in h^{-1}(0)$ and $Z_2 f \subset Z_2 g$ for $g \in Lgu(X)$, then $g \in h^{-1}(0)$. Suppose $g \notin h^{-1}(0)$ and $h(g) = a \neq 0$. Then $g - \bar{a} \in h^{-1}(0)$ and $Z_2(g - \bar{a}) \in Z_2 h^{-1}(0)$. From this we obtain a contradiction as $Z_2 f \cap Z_2(g - \bar{a}) \subset Z_2 g \cap Z_2(g - \bar{a}) = \varnothing$. Having shown that

$Z_2h^{-1}(0)$ is a filter, suppose there exists a filter ψ in Z_2X such that $Z_2h^{-1}(0) \subsetneq \psi$. Let $Z_2g \in \psi \backslash Z_2h^{-1}(0)$. Then $h(g) = a \neq 0$, and we obtain again that $g - \bar{a} \in h^{-1}(0)$, $Z_2(g - \bar{a}) \in Z_2h^{-1}(0) \subset \psi$ and $\emptyset = Z_2g \cap Z_2(g - \bar{a}) \in \psi$. This contradiction shows that $Z_2h^{-1}(0)$ is an ultrafilter.

LEMMA *13: Let* $X \in$ **COTS** *and* $h{:}Lgu(X) \to \boldsymbol{R}$ *be a surjective* **PLG**-*homomorphism. There exists a unique* $x \in X$ *such that* $h(f) = f(x)$ *for all* $f \in Lgu(X)$.

Proof: We have just shown that $Z_2h^{-1}(0)$ is an ultrafilter in Z_2X. Since X is in **COTS**, and is therefore locally compact and normally ordered with continuous order, we conclude as in Lemma 7 that $\bigcap Z_2h^{-1}(0)$ has at most one element. Since X is compact, it follows by the finite intersection property of $Z_2h^{-1}(0)$ that there exists $x \in \bigcap Z_2h^{-1}(0)$, and accordingly $\bigcap Z_2h^{-1}(0) = \{x\}$.

Let $f \in Lgu(X)$, and $h(f) = b$. Then $h(f - \bar{b}) = 0$, and $f - \bar{b} \in h^{-1}(0)$ follows. Therefore $x \in \bigcap Z_2h^{-1}(0) \subset Z_2(f - \bar{b})$ and we obtain $0 = (f - \bar{b})(x) = f(x) - b$, which means $h(f) = b = f(x)$.

THEOREM *11*: *If* X, $Y \in$ **COTS**, *and* $\psi{:}(Lgu(y), C_1Y) \to (Lgu(X), C_1X)$ *is a* **p-PLG**-*isomophism there exists* $f{:}X \to Y$, *a* **COTS**-*isomorphism such that* $\psi = C_1(f)$.

Proof: The proof is essentially as in Theorem 3, using here Lemma 13, to apply Theorem 1.

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REFERENCES

[1] T. H. CHOE AND O. C. GARCÍA, *Epireflective subcategories of partially ordered topological spaces*. Kyungpook Math. J. **13**(1973) 97–107.
[2] O. C. GARCÍA, Colimits and coreflective subcategories in partially ordered topological spaces. Bol. Soc. Mat. Mexicana, **18**(1973) 72–81.
[3] L. GILLMAN AND M. JERISON, Rings of Continuous Functions, van Nostrand, Princeton, 1960.
[4] E. HEWITT, *Rings of real valued continuous functions*, Trans. Amer. Math. Soc. **64**(1948) 45–99.
[5] I. KAPLANSKY, *Lattices of continuous functions I*. Bull. Amer. Math. Soc. **53**(1947) 617–23.
[6] A. N. MILGRAM, *Multiplicative semigroups of continuous functions*, Duke Math. J. **16**(1949) 377–83.
[7] L. NACHBIN, Topology and Order, van Nostrand, Princeton, 1965.
[8] B. PAREGIS, Categories and Functors, Academic Press, New York-London, 1970.
[9] T. SHIROTA, *A class of topological spaces*, Osaka J. Math. **4**(1952) 23–39.
[10] ———, *A generalization of a theorem by Kaplansky*, Osaka J. Math. **4**(1952) 121–32.
[11] M. H. STONE, *Applications of the theory of Boolean rings to general topology*. Trans. Amer. Math. Soc. **41**(1937) 1–25.