THE PHRAGMEN-BROUWER THEOREM FOR SEPARATED SETS

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1. Introduction

In this paper we prove the following result, which we call the *Phragmen-Brouwer* theorem for separated sets.

Theorem. *In a connected locally connected space X the following properties are equivalent*:

(i) *X is unicoherent,*

(ii) *if M, N are separated sets whose union separates two points p, q, then M or N separates p, q,*

(iii) *if M, N are separated sets whose union separates X, then M or N separates X.*

The case of this theorem in which M, N are disjoint closed sets is well-known, for it is proved on pp. 47–49 of [12]. (It is also partially proved in theorem 1 of [11].) It is a corollary of this special case that the three properties are equivalent if X is in addition a completely normal space. From this point of view our theorem states that the equivalences continue to hold even when X satisfies no separation axioms at all.

In §2 we give the definitions and notation used in the paper. In §3 we give the complete proof of the theorem. However, the initial parts of the proof are presented in four lemmas. It will be noticed that the statement of the theorem does not explicitly present us with a pair of disjoint closed sets with which to start working. In lemma 1 we remedy this situation by showing how a pair of disjoint closed sets can be constructed in an arbitrary space having two pairs of separated sets whose unions are complementary. In lemma 2 we collect several simple properties of connected locally connected unicoherent spaces. This enables us in lemma 3 to relate the construction of lemma 1 more closely to the context of the theorem. Lemma 4 is a quotation of the principal lemma of [3], which is needed in the final stage of the proof. We then give the proof of the theorem itself, dividing it into several cases. In a corollary to the theorem, we generalize Stone's theorem on "open-unicoherence" in [11]. We close §3 with a short proof that (i) implies (iii) in the theorem.

In §4 we raise a question which is related to the theorem, and in §5 we add some remarks indicating how the nomenclature "Phragmen-Bouwer theorem" has been used in the literature.

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2. Definitions and Notation

All sets mentioned are subsets of an arbitrary space X.

A space X is said to be *unicoherent* if for each pair of connected closed sets M, N such that $X = M \cup N$, $M \cap N$ is connected.

We say that M, N are *separated* sets if $M \cap \bar{N} = \phi = \bar{M} \cap N$. A set L *separates the points* p, q if there are separated sets M, N containing p, q, respectively, such that $X - L = M \cup N$. A set L *separates* X if it separates some pair of points.

As a matter of notation in set theory, we take "$\cup$", "$\cap$" to have precedence over "$-$"; thus, for example, $X - M \cup N$ means $X - (M \cup N)$. We denote the frontier of a set L by $Fr\, L$; i.e., $Fr\, L = \bar{L} \cap \overline{X - L}$. We also introduce the following special notation: if L, C are arbitrary sets, we denote by L_C the union of all the components of $X - C$ whose frontiers are contained in L. The notation $L = [M] \cup [N]$ means that $L = M \cup N$ and M, N are separated sets. Finally, the phrase $L = [M] \cup [N]$ *is a separation* means in addition that $M \neq \phi \neq N$.

3. Proof of the Theorem

In the first lemma we show how, in an arbitrary space X having two pairs of separated sets whose unions are complementary, a pair of disjoint closed sets can easily be constructed, and we give its needed properties.

LEMMA 1. *Let* (M, N), (P, Q) *be pairs of separated sets in an arbitrary space* X *whose unions are complementary. Then* $A = \bar{Q} \cap \bar{M}$, $B = \bar{P} \cap \bar{N}$ *are disjoint closed sets contained in* $Q \cup M$, $P \cup N$, *respectively; moreover, every component* C *of* $X - A \cup B$ *is contained in either* $P \cup M$ *or* $Q \cup N$.

Proof. By definition A, B are closed sets. In order to prove that they are disjoint observe that $\bar{M} \cap \bar{N} \subset P \cup Q$, because M, N are separated, and $\bar{P} \cap \bar{Q} \subset M \cup N$ because P, Q are separated. Thus

$$\begin{aligned} A \cap B &= (\bar{Q} \cap \bar{M}) \cap (\bar{P} \cap \bar{N}), \\ &= (\bar{P} \cap \bar{Q}) \cap (\bar{M} \cap \bar{N}), \\ &\subset (M \cup N) \cap (P \cup Q), \\ &\subset \phi; \end{aligned}$$

i.e., A, B are disjoint. Since $\bar{M} \cap N = \phi = P \cap \bar{Q}$

$$A \cap N = (\bar{Q} \cap \bar{M}) \cap N = \bar{Q} \cap (\bar{M} \cap N) = \phi,$$

$$A \cap P = (\bar{Q} \cap \bar{M}) \cap P = (P \cap \bar{Q}) \cap \bar{M} = \phi.$$

Thus $A \subset Q \cup M$. Similarly $B \subset P \cup N$.

Now let C be an arbitrary component of $X - A \cup B$. We show that C is contained in $P \cup M$ or in $Q \cup N$ as follows. Since P, Q are separated,

$$\begin{aligned} \bar{P} \cap \bar{Q} &\subset M \cup N, \\ &\subset (\bar{M} \cup \bar{N}) \cap (\bar{P} \cap \bar{Q}), \\ &\subset (\bar{M} \cap \bar{P} \cap \bar{Q}) \cup (\bar{N} \cap \bar{P} \cap \bar{Q}), \\ &\subset (\bar{Q} \cap \bar{M}) \cup (\bar{P} \cap \bar{N}). \end{aligned}$$

Similarly, since M, N are separated, we deduce that

$$\bar{M} \cap \bar{N} \subset (\bar{Q} \cap \bar{M}) \cup (\bar{P} \cap \bar{N}).$$

Thus

$$\begin{aligned} (\bar{Q} \cup \bar{N}) \cap (\bar{P} \cup \bar{M}) &= (\bar{P} \cap \bar{Q}) \cup (\bar{Q} \cap \bar{M}) \cup (\bar{P} \cap \bar{N}) \cup (\bar{M} \cap \bar{N}), \\ &= (\bar{Q} \cap \bar{M}) \cup (\bar{P} \cap \bar{N}), \\ &= A \cup B; \end{aligned}$$

therefore

$$\begin{aligned} X - A \cup B &= X - (\bar{Q} \cup \bar{N}) \cap (\bar{P} \cup \bar{M}), \\ &= (X - \bar{Q} \cup \bar{N}) \cup (X - \bar{P} \cup \bar{M}). \end{aligned}$$

Since $X - \bar{Q} \cup \bar{N}, X - \bar{P} \cup \bar{M}$ are disjoint open sets, it follows from the last identity that C is contained in one of them. However, $X - \bar{Q} \cup \bar{N}, X - \bar{P} \cup \bar{M}$ are contained in $P \cup M, Q \cup N$, respectively, because $X = M \cup N \cup P \cup Q$. Thus C is contained in $P \cup M$ or in $Q \cup N$.

In the next lemma we collect several simple properties of connected locally connected unicoherent spaces.

LEMMA 2. *Let A, B be disjoint closed sets in a connected locally connected unicoherent space X, and let C be a component of $X - A \cup B$. Then A_C, B_C are disjoint closed sets such that $X - C = A_C \cup B_C$, and $Fr\, A_C = A \cap \bar{C}$, $Fr\, B_C = B \cap \bar{C}$.*

Further, if E, F are the components of $X - B$, $X - A$, respectively, which contain C, then $E - C \subset A_C$, $F - C \subset B_C$,

Proof. Let D be an arbitrary component of $X - C$. We denote by (*) the proposition that either $Fr\, D \subset A \cap \bar{C}$ or $Fr\, D \subset B \cap \bar{C}$, and we proceed to prove it. It is a consequence of the connectedness of the space that $Fr\, D \neq \phi$, and of the local connectedness of the space that $Fr\, D \subset Fr\, C$. This inclusion and the unicoherence of the space imply that $Fr\, D$ is connected, by theorem 1 (iii) of [11]. Since $Fr\, C \subset A \cup B$ and A, B are disjoint closed sets, it follows that $Fr\, D \subset A \cap \bar{C}$ or $Fr\, D \subset B \cap \bar{C}$. The two possibilities are exclusive, because $Fr\, D \neq \phi$.

It is an immediate consequence of (*) that A_C, B_C are disjoint sets such that $X - C = A_C \cup B_C$. In order to show that A_C is closed, it is certainly sufficient to show that

$$Fr\, A_C \subset A \cap \bar{C} \subset A_C.$$

For this purpose, let $\{D_\alpha\}_\alpha$ be the collection of components of A_C. Then by theorem 1, p. 236 of [7] and (*).

$$Fr\, A_C \subset \overline{\bigcup_\alpha Fr\, D_\alpha},$$
$$\subset A \cap \bar{C},$$

On the other hand, take $x \in A \cap \bar{C}$. Then x belongs to a component D of $X - C$, and since $Fr\, D$ meets $A \cap \bar{C}$ at least in the point x, it follows from (*) that $Fr\, D \subset A \cap \bar{C}$. That is, $Fr\, D \subset A$, which means that D is a component of A_C. In particular, $x \in A_C$. Thus $A \cap \bar{C} \subset A_C$. Similarly one shows that $Fr\, B_C \subset B \cap \bar{C} \subset B_C$, which implies that B_C is closed.

That $Fr\, A_C = A \cap \bar{C}$ is an immediate consequence of the relations $Fr\, A_C \subset A \cap \bar{C} \subset A_C$, because C lies in the complement of A_C. Similarly $Fr\, B_C = B \cap \bar{C}$.

Finally, suppose that $E - C \not\subset A_C$. Then the relation $X - C = A_C \cup B_C$, which we have established, implies that a component D of B_C meets $E - C$. By definition $Fr\, D \subset B$, and this implies that $E \cap Fr\, D = \phi$, because E is a component of $X - B$. Thus $D \cap E$ is a relatively open and closed subset of the subspace E, and $D \cap E \neq \phi \neq C \subset E - D$. This contradiction to the connectedness of E shows that $E - C \subset A_C$. Similarly one shows that $F - C \subset B_C$.

The proof of the theorem is largely based on the identities in the following lemma.

LEMMA 3. *Let (M, N), (P, Q) be pairs of separated sets in an arbitrary space X whose unions are complementary, so that we may write*

$$X - M \cup N = [P] \cup [Q].$$

Let $A = \bar{Q} \cap \bar{M}, B = \bar{P} \cap \bar{N}$, and let C be a component of $X - A \cup B$. Then

$$X - M = [A_C - M] \cup [B_C \cup C - M]$$

or

$$X - N = [A_C \cup C - N] \cup [B_C - N],$$

and the first of these alternatives holds if $C \subset P \cup M$, while the second holds if $C \subset Q \cup N$.

Proof. Every component of $X - A \cup B$ is contained in either $P \cup M$ or $Q \cup N$, by lemma 1.

Thus, supposing first that $C \subset P \cup M$, we shall show that

$$X - M = [A_C - M] \cup [B_C \cup C - M].$$

This is valid as a set identity because $X - C = A_C \cup B_C$, by lemma 2. In order to show that $A_C - M$, $B_C \cup C - M$ are separated, it is enough to show that $A_C - M$, $C - M$ are separated, because A_C, B_C are disjoint closed sets, by lemma 2. The closedness of A_C implies that

$$\overline{A_C - M} \cap (C - M) \subset A_C \cap C,$$
$$\subset \phi.$$

The relation $A_C \cap C = \phi$ implies that $A_C \cap \bar{C} \subset Fr\, A_C$. That is, $A_C \cap \bar{C} \subset A$, by lemma 2, and consequently

$$(A_C - M) \cap \bar{C} \subset A - M,$$

and
$$(A_C - M) \cap \overline{C - M} \subset (A - M) \cap \overline{C - M}.$$

But $A \subset Q \cup M$ by lemma 1, and we are supposing that $C \subset P \cup M$, so

$$(A - M) \cap \overline{C - M} \subset Q \cap \bar{P},$$
$$\subset \phi.$$

because P, Q are separated. This proves that $A_C - M$, $C - M$ are separated.

If $C \subset Q \cup N$, a similar argument shows that

$$X - N = [A_C \cup C - N] \cup [B_C - N].$$

Indeed, we have only to interchange the letters M and N, P and Q, and A and B in the previous paragraph.

For convenience we quote the principal lemma of [3]. The notation is selected to conform to the usage in the proof of the theorem that follows.

Lemma 4. *Let U be an open set in a connected locally connected unicoherent space X such that there is a separation*

$$X - U = [A'] \cup [B'],$$

in which A', B' contain given points q, p, respectively. Then there is a component C of U such that the components G, H of $X - C$ which contain p, q, respectively, satisfy either

$$Fr\, G \subset Fr\, A' \quad \textit{and} \quad Fr\, H \subset Fr\, B'$$

or
$$Fr\, G \subset Fr\, B' \quad \textit{and} \quad Fr\, H \subset Fr\, A'.$$

Now we prove the theorem which is stated in §1.

Proof of Theorem. We need only show that (i) implies (ii), for (ii) implies (iii) trivially, and if (iii) holds then, in particular, whenever the union of a pair of disjoint closed sets separates X, one of them does. This is the Phragmen-Brouwer property of p. 47 of [12], and in p. 48, 49 of [12] it is shown that this implies that X is unicoherent; i.e., (iii) implies (i).

Thus, in order to show that (i) implies (ii), let X be a connected locally connected unicoherent space, and let M, N be separated sets such that there is a separation

$$X - M \cup N = [P] \cup [Q],$$

where P, Q contain the given points p, q, respectively. Also put $A = \bar{Q} \cap \bar{M}$, $B = \bar{P} \cap \bar{N}$, as in lemma 1.

If $A = \phi$, then Q, M are separated so that

$$X - N = [P \cup M] \cup [Q]$$

and N separates p, q, while if $B = \phi$ then P, N are separated so that

$$X - M = [P] \cup [Q \cup N]$$

and M separates p, q. Thus we shall suppose that $A \neq \phi \neq B$.

Provided that $p \notin B$, let E be the component of $X - B$ that contains p. Otherwise let $E = \phi$. Similarly, provided that $q \notin A$, let F be the component of $X - A$ that contains q. Otherwise let $F = \phi$. We divide the proof into the four cases (I) $p, q \in E$, (II) $p, q \in F$, (III) $q \notin E$, $p \notin F$ and $E \cap F \neq \phi$, and (IV) $q \notin E, p \notin F$ and $E \cap F = \phi$. These four cases cover all possibilities.† In fact, if (I) does not hold, then $q \notin E$ or $p \notin E$, and this implies that $q \notin E$, for if $p \notin E$ then $E = \phi$ by definition. Similarly, if (II) does not hold then $p \notin F$. Thus the negation of the disjunction of (I), (II) implies the disjunction of (III), (IV).

Case (I): $p, q \in E$. In this case there is a component C of $E - A$ which contains p, because $A \subset Q \cup M$ by lemma 1. Also C is a component of $X - A \cup B$. In fact the components of $X - B$ other than E are open sets in the complement of C, as are the components of $E - A$ other than C. Thus $\bar{C} \subset A \cup B \cup C$. Since C is in addition a connected open subset of $X - A \cup B$, it follows that C is a component of $X - A \cup B$. Since $p \in C$, it follows from lemma 1 that $C \subset P \cup M$. Thus

$$X - M = [A_C - M] \cup [B_C \cup C - M]$$

by lemma 3. In order to see that $q \in A_C - M$, notice that C is a component of $X - A \cup B$ and E is the component of $X - B$ in which it lies. Thus $E - C \subset A_C$ by lemma 2. But $q \in E - C$, because $C \subset P \cup M$. Consequently $q \in A_C - M$. Since $p \in B_C \cup C - M$, this shows that M separates p, q.

Case (II): $p, q \in F$. The argument in this case is identical to that in the previous one if we interchange the letters p and q, M and N, A and B, and E and F in it.

Case (III): $q \notin E, p \notin F$ and $E \cap F \neq \phi$. In this case let $C = E \cap F$. We show that C is a component of $X - A \cup B$. Since E, F are connected open sets with disjoint frontiers (for their frontiers are contained in A, B, respectively, which are disjoint by lemma 1), it follows from the unicoherence of X and theorem 1 (iv) of [11] that $E \cap F$ is connected. Thus C is a non-empty connected subset of $X - A \cup B$. As such C lies in a well-defined component D of $X - A \cup B$, but D in turn lies in both E and F. That is, $C = D$, and so C is itself a component of $X - A \cup B$. Thus

$$X - M = [A_C - M] \cup [B_C \cup C - M]$$

or

$$X - N = [A_C \cup C - N] \cup [B_C - N]$$

† Incidentally, the four cases are also mutually exclusive. At the beginning of the proof of case (III) it is shown that $E \cap F$ is either empty or a component of $X - A \cup B$. It follows from lemma 1 that $E \cap F$ is contained either in $P \cup M$ or in $Q \cup N$. Thus $p, q \in E \cap F$ is impossible, which shows that (I), (II) are exclusive. Also (III), (IV) are exclusive and each implies the negation of (I) and the negation of (II).

by lemma 3. Now notice that C is a component of $X - A \cup B$ and E, F are the components of $X - B, X - A$, respectively, in which it lies. Thus $E - C \subset A_C$, $F - C \subset B_C$ by lemma 2. Also $p \in E, q \in F$, because $E \cap F \neq \phi$ by hypothesis, and $p, q \notin C$ because $q \notin E, p \notin F$ by hypothesis. Thus $p \in A_C$, $q \in B_C$. This suffices to show that M separates p, q if the first of the two displayed alternatives holds, and that N separates p, q if the second holds.

Case (IV): $q \notin E, p \notin F$ and $E \cap F = \phi$. In this case define

$$A' = (A - E) \cup F,$$

or

$$B' = (B - F) \cup E.$$

Then A', B' are disjoint closed sets containing q, p, respectively, and $Fr\, A' \subset A$, $Fr\, B' \subset B$ and $A \cup B \subset A' \cup B'$.

We first prove these simple properties. That A', B' are disjoint is a consequence of the disjointness of A, B, established in lemma 1, and the hypothesis $E \cap F = \phi$. In order to see that A' is closed, observe that, since F is either empty or a component of $X - A$, $\bar{F} \subset A \cup F$. Since E is an open set which does not meet F by hypothesis, this implies that $\bar{F} \subset (A - E) \cup F$. Thus $A' = (A - E) \cup \bar{F}$, and this shows that A' is closed, because $A - E$ is the difference between a closed set and an open set. A similar argument shows that B' is closed. Next, $q \in A \cup F$ by the definition of F, and $q \notin E$ by hypothesis; i.e., $q \in A'$. Similarly $p \in B'$. Next, $Fr\, A' \subset Fr\, (A - E) \cup Fr\, F$, and $Fr\, (A - E) \subset A$ because A is closed, while $Fr\, F \subset A$ by the local connectedness of the space; i.e., $Fr\, A' \subset A$. Similarly, $Fr\, B' \subset B$. Finally, $A \cup B \subset A' \cup B'$ because $A \subset (A - E) \cup E$ and $B \subset (B - F) \cup F$.

Now define U as the complement of the closed set $A' \cup B'$, so that

$$X - U = [A'] \cup [B']$$

is a separation in which $q \in A', p \in B'$. Then by lemma 4 there is a component C of U such that the components G, H of $X - C$ which contain p, q, respectively, satisfy either the relations $Fr\, G \subset Fr\, A'$ and $Fr\, H \subset Fr\, B'$, or the relations $Fr\, G \subset Fr\, B'$ and $Fr\, H \subset Fr\, A'$. However, $Fr\, A' \subset A$, $Fr\, B' \subset B$ by lemma 2, so we can say that either

$$Fr\, G \subset A \quad \textit{and} \quad Fr\, H \subset B \cdots (\alpha)$$

or

$$Fr\, G \subset B \quad \textit{and} \quad Fr\, H \subset A \cdots (\beta)$$

We show that C is a component of $X - A \cup B$. Since all the components of U are open sets, it follows that $Fr\, C \subset Fr\, U$. But $Fr\, U \subset Fr\, A' \cup Fr\, B'$ and we have already shown that $Fr\, A' \subset A$, $Fr\, B' \subset B$. Thus $Fr\, C \subset A \cup B$. But we have also shown that $A \cup B \subset A' \cup B'$, so that C is in addition a connected open subset of $X - A \cup B$. This implies that C is a component of $X - A \cup B$. Thus

$$X - M = [A_C - M] \cup [B_C \cup C - M],$$

or

$$X - N = [A_C \cup C - N] \cup [B_C - N]$$

by lemma 3. Moreover, if (α) holds then G, H are components of A_C, B_C, respectively, so that $p \in A_C$, $q \in B_C$, while if (β) holds then G, H are components of B_C, A_C, respectively, so that $p \in B_C$, $q \in A_C$. This suffices to show that M separates p, q if the first of the two displayed alternatives holds, and that N separates p, q if the second holds.

As a matter of interest we mention the following corollary, because the property which it shows characterizes unicoherence in connected locally connected spaces includes both the property that is used to define unicoherence and the property that is used to define "open-unicoherence" on p. 432 of [11]. Thus, in particular, the corollary subsumes theorem 3 of [11].

Corollary. *If X is a connected locally connected space, then X is unicoherent if and only if it satisfies the following property: if K, L are connected sets such that $X = K \cup L$ and $X - L$, $X - K$ are separated sets, then $K \cap L$ is connected.*

Proof. Let X be unicoherent and suppose on the contrary that there are connected sets K, L such that $X = K \cup L$ and $X - L$, $X - K$ are separated sets, but $K \cap L$ is not connected. Let $K \cap L = [P] \cup [Q]$ be a separation, and put $M = X - L$, $N = X - K$. Then M, N are separated sets and

$$X - M \cup N = [P] \cup [Q]$$

is a separation. It follows from property (iii) in the theorem that M separates X or N separates X. However, the complements of M, N are the connected sets L, K, respectively, which is a contradiction.

The converse is obvious, for the given property implies that each pair of connected closed sets whose union is X has a connected intersection.

The proof of this corollary uses only the part of the theorem stating that (i) implies (iii). There is a short proof of this implication, which bypasses the need for going through the circular argument as given in the proof of the theorem. We give the short proof that (i) implies (iii) in the next paragraph.

Let X be a connected locally connected unicoherent space and M, N two separated sets such that there is a separation

$$X - M \cup N = [P] \cup [Q],$$

and put $A = \bar{Q} \cap \bar{M}$, $B = \bar{P} \cap \bar{N}$. As at the beginning of the proof of the theorem, if $A = \phi$ then N separates X and if $B = \phi$ then M separates X. So suppose that $A \neq \phi \neq B$. Since A, B are disjoint closed sets by lemma 1, it follows from the connectedness and local connectedness of the space that there is a component C of $X - A \cup B$ such that $A \cap \bar{C} \neq \phi \neq B \cap \bar{C}$. Also $C \subset P \cup M$ or $C \subset Q \cup N$, by lemma 1. Suppose first that $C \subset P \cup M$. Then

$$X - M = [A_C - M] \cup [B_C \cup C - M]$$

by lemma 3. In order to show that M separates X it must be shown that $A_C - M \neq \phi \neq B_C \cup C - M$. In fact $A_C \cap Q \neq \phi$. For suppose on the contrary that $A_C \cap Q = \phi$. Then $Q \subset B_C$, because $X - C = A_C \cup B_C$ by lemma 2 and $C \subset$

$P \cup M$ by hypothesis. That is, $\bar{Q} \subset B_C$, because B_C is closed by lemma 2. Since $A = \bar{Q} \cap \bar{M}$, it follows that $A \cap \bar{C} \subset B_C$. However, $A \cap \bar{C} \subset A_C$ as well, because, by lemma 2, $Fr\ A_C = A \cap \bar{C}$ and A_C is closed. Thus $A_C \cap B_C \neq \phi$, because $A \cap \bar{C} \neq \phi$. This contradiction to lemma 2 shows that $A_C \cap Q \neq \phi$. Consequently $A_C - M \neq \phi$. On the other hand, the relation $B \subset P \cup N$ in lemma 1 implies that $(B \cap \bar{C}) \cap M = \phi$. Since $B \cap \bar{C} \subset B_C$ by lemma 2, it follows that $B \cap \bar{C}$ is a non-empty subset of $B_C \cup C - M$. This shows that M separates X. Similarly, one shows that if $C \subset Q \cup N$, then N separates X.

4. A Question

In this section we raise a question an affirmative answer to which would give the theorem of this paper as an immediate corollary. We also present the evidence for expecting such an answer.

Question. Are the following properties equivalent in a connected locally connected space X:

(i)′ X is unicoherent,

(ii)′ if L is an arbitrary set which separates two points p, q, then a component of L separates p, q,

(iii)′ if L is an arbitrary set which separates X, then a component of L separates X?

It is well-known that these properties are equivalent when L is a closed set. This case is partially proved in theorem 1 of [11], and the part that is missing can easily be supplied using standard arguments on connected locally connected spaces. It is a corollary of this special case that the three properties are equivalent if X is in addition a completely normal space.

It has recently been shown in [3] that the three properties are equivalent when L is an open set. Our reason for raising the question here, however, is that it follows from the theorem of this paper that the three properties are also equivalent when L is an arbitrary set with a finite number of components.

5. Historical Note

In each of the papers [8], [5], [6], [1], [4], [13] the term "Phragmen-Brouwer theorem" is used to denote a certain result, but it does not designate the same result in all cases. Thus we add some remarks on the papers [9], [10] of Phragmen and [2] of Brouwer from which the nomenclature arises.

In [9] Phragmen proved that (a) if P is a closed subset of the plane E^2 having no non-degenerate connected subset, then $E^2 - P$ is connected. In order to make this result more accessible, Mittag-Leffler suggested that it be reproduced in Acta Mathematica. This was done in [10], where Phragmen reformulated the result and proved instead that (b) if A is a connected open subset of the plane E^2 and $E^2 - \bar{A} \neq \phi$, then $Fr\ A$ contains a non-degenerate connected set. Phragmen was under the impression that this was a generalization of his original result, but it is easily seen that the properties ascribed to the plane in (a), (b) are equivalent in any locally connected space.

In [2] Brouwer gave a proof of the Jordan curve theorem, a step towards which was the result that (c) if K is a compact connected set in the plane E^2 and C is a component of $E^2 - K$, then $Fr\, C$ is connected. By using a suitably chosen inversion of the plane, it is easily seen that (c) implies (b).

In [5], [6] the result (c) is called *Brouwer's* theorem, but in [8] it is called the *Phragmen-Brouwer* theorem. On the other hand, in [1], [4], [13] the *Phragmen-Brouwer* theorem denotes something else, namely the result that (d) if M, N are disjoint bounded closed subsets of the plane E^2 whose union separates E^2, then M or N separates E^2. However, using a suitably chosen inversion of the plane and some standard arguments on local connectedness, it is easily shown that (c), (d) are equivalent.

Notwithstanding these ambiguities, the source from which the term "Phragmen-Brouwer" is most familiar is probably [12]. On p. 47 of this book an arbitrary connected locally connected space X having property (iii) of the theorem of this paper for the special case in which M, N are disjoint closed sets is said to have the *Phragmen-Brouwer* property. Since this usage is an extension of the term as used in [1], [4], [13], we have in turn extended it to the theorem of this paper.

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