A NOTE ON FOLIATIONS ON SPHERES

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Here we prove:

Theorem 1. *Any compact, connected, smooth 7-manifold which fibres differentiably over S^4, admits a smooth codimension three foliation.*

Theorem 2. *Any compact, connected, smooth 15-manifold which fibres over S^8 with fibre S^7 admits a codimension seven smooth foliation.*

Corollary 1. *There exists a codimension three smooth foliation of S^7.*

Corollary 2. *There exists a codimension seven smooth foliation on S^{15}*

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Proof of Theorem 1. Let us decompose S^4 as $D^2 \times S^2 \cup_f S^1 \times D^3$. That is to say, let us consider S^4 as obtained from $A = D^2 \times S^2$ and $B = S^1 \times D^3$ by "gluing" differentiably their boundaries through the diffeomorphism $f: S^1 \times S^2 \to S^1 \times S^2$.

Let $\pi: M^7 \to S^4$ be a smooth fibration from the compact, connected, smooth 7-manifold M^7 over S^4 with typical fibre diffeomorphic to S^3. Since the fibration over both A and B is trivial, M^7 is obtained from $C = D^2 \times S^2 \times S^3$ and $G = S^1 \times D^3 \times S^3$ by "gluing" differentiably their boundaries by means of a diffeomorphism

$$F: S^1 \times S^2 \times S^3 \to S^1 \times S^2 \times S^3,$$

where $F(\theta, x, y) = (\theta, x, \varphi(\theta, x)(y))$, for some map $\varphi: S^1 \times S^2 \to \mathrm{Diff}^\infty(S^3)$ for which the evaluation map is smooth.

Let G be foliated by the inverse images given by the projection $p: S^1 \times D^3 \times S^3 \to D^3$.

Then, G is saturated by leaves of the form

$$S^1 \times \{t\} \times S^3,$$

with $t \in \partial D^3 = S^2$.

Now, by [1], there exists a codimension one folitation, $\mathfrak{F}$, of $D^2 \times S^3$ such that the boundary is saturated by leaves. Furthermore, we can assume that a collar neighborhood of $D^2 \times S^3$, of the form $V = \{(te^{i\theta}, x) \in D^2 \times S^3 : \frac{1}{2} \leq t \leq 1\}$ is saturated by leaves of the form

$$L_t = \{(te^{i\theta}, x) : 0 < \theta \leq 2\pi, x \in S^3\}$$

with $\frac{1}{2} \leq t \leq 1$. Therefore, we can foliate C, in codimension 3, by considering as leaves the inverse images of the leaves of $\mathfrak{F}$ under the projection onto the second factor $p_1:D^2 \times S^2 \times S^3 \to S^2$. Then, it is easy to see, using the fact that F preserves the second factor, that we can glue, differentiably, C with G along their boundaries in such a way that their respective foliations match differentiably at their boundaries and in such a way that we obtain a manifold diffeomorphic with M^7.

Theorem 2 can be proved by a construction analogous to the one given above, because

$$M^{15} = D^2 \times S^6 \times S^7 \cup_F S^1 \times D^7 \times S^7$$

where $F: S^1 \times S^6 \times S^7 \to S^1 \times S^6 \times S^7$ is a diffeomorphism of the form

$$F(x, y, z) = (x, y, \varphi(x, y)z),$$

and also because there exists a codimension one foliation of $D^2 \times S^7$ such that the boundary is foliated by leaves of the type $\partial D^2 \times S^7$.

CENTRO DE INVESTIGACION DEL IPN

REFERENCE

[1] H. B. LAWSON, *Codimension-one foliations of spheres*, Ann. Math. **94** (1971), 494–503.