RELATIVE PRINCIPAL FIBRATIONS

BY J. F. McCLENDON

1. Introduction

A relative principal fibration is, roughly speaking, a fibration in a category that is induced from a principal fibration in a subcategory. A better name might be co-relative principal fibration since the notion is relative to an object under a given object rather than to a subobject. It was shown in [5] that relative principal fibrations arise naturally in the factorization of non-orientable fibrations. It will be shown in a separate paper (Reducing Towers of Principal Fibrations) that the class of relative principal fibrations is fairly large. In particular, if $F \to E \to B$ is a fibration with F n-connected and $\pi_j(F) = 0$, $i > 2n$, then it is a relative principal fibration (no connectivity assumptions on B). The exact sequence of the present paper is essential to the proof of that result. It should be noted that even in the simple case of principal fibrations in the category of pointed spaces and maps the results here, while not unknown, are more general than the familiar path-loop sequence results.

Section 2 discusses some properties of the category $\text{Top}(C \to D)$. Functors $\hat{\Omega}$ and $\hat{\Sigma}$ are introduced and some adjoint equations are noted. In section 3 "relative principal fibration" is defined and the basic lifting properties are proved. Section 4 describes a long exact sequence extending part of the results of section 3.

There is a notion of relative principal cofibration. There objects are being studied and applied by D. Kruse.

2. Basic Results

Let $\text{Top}(u:C \to D)$ be the category whose objects are triples $(X, \hat{x}, \check{x})$ with $\check{x}:C \to X$, $\hat{x}:X \to D$ and $\hat{x}\check{x} = u$. The maps are maps $f:X \to Y$ with $f\check{x} = \check{y}$ and $\hat{y}f = \hat{x}$. The main properties of $\text{Top}(C \to D)$ were noted in [4] (see [3] for more detail). Denote $\text{Top}(id:D \to D)$ by $\text{Top}(D = D)$. Recall that it has the essential homotopy properties of $\text{Top}(pt = pt)$, the category of pointed spaces and maps. The idea is that the fibers $\hat{x}^{-1}(d)$ (but no fibration assumptions on $\hat{x}$ are made) have a natural base point $\hat{x}^{-1}d \cap \check{x}D$ and standard homotopy can be done there. One can also view D as a parameter space and think of doing ordinary homotopy with a parameter d. Many of the basic functors can be recovered from the following definition.

2.1 *Definition.* Let a map $\hat{y}:Y \to D$ and a space W be given and $W_0 \subset W$, $Y_0 \subset Y$. Define $F(W, W_0 ; Y, Y_0 \mid \hat{y}) = \{w:(W, W_0) \to (Y, Y_0) \mid \hat{y}w \text{ is constant}\}$ Give it the compact-open topology.

2.2 *Notes.* (1) We have $F \to D$, $w \to$ the constant value of $\hat{y}w$. Denote this map by $\hat{y}$ also or $\hat{y}'$ if confusion seems likely. If $\check{y}:C \to Y_0$ is given with $\hat{y}\check{y} = u$ then define $C \to F$ by $c \to w$, $w(W) = \check{y}c$; denote this map by $\check{y}$.

(2) If $Y \in \mathrm{Top}(D = D)$ then

$$P_D Y = F((I, 0); (Y, \check{y}D) \mid \hat{y})$$
$$\Omega_D Y = F((S^1, 1); (Y, \check{y}D) \mid \hat{y})$$

are the path and loop functors for $\mathrm{Top}(D = D)$ (see [4]).

(3) For $Y \in \mathrm{Top}(C \to D)$ define $\hat{\Omega}_D{}^n Y = F(S^n, Y \mid \hat{y})$.

This can be viewed as an object in $\mathrm{Top}(C \to D)$. However, it can also be viewed as an object in $\mathrm{Top}(Y = Y)$ as follows. Let p be the base point of S^n, $p = (1, 0, 0, \cdots, 0)$ and let $\hat{\Omega}_D{}^n Y \to Y$ be $w \to w(p)$. Let $Y \to \Omega_D{}^n Y$ send y to the constant function at y. The iteration formula suggested by the notation isn't true—but the following one is.

2.3 Lemma.

$$\hat{\Omega}_D{}^j Y = \Omega_Y{}^{j-i} \hat{\Omega}_D{}^i Y \qquad 0 \le i \le j.$$

(4) Let a space A and a map $\check{x}: C \to X$ be given. Then the dual to 2.1 is $A \times X/R$ where R is the equivalence relation generated by $(a, x) \sim (a', x)$ all $a, a' \in A$ whenever $x \in \check{x}C$. In particular we obtain

$$\check{\Sigma}_n{}^C X = S^n \times X/R.$$

(5) The functors $\hat{\Omega}$ and $\check{\Sigma}$ are related by several different adjointness equations.

(a) Suppose $X, Y \in \mathrm{Top}(C \to \mathrm{R})$. Then we have

$$[\check{\Sigma}_n{}^C X, Y]_D{}^C = [X, \hat{\Omega}_D{}^n Y]_D{}^C$$

(b) Suppose a fixed map $X \to Y$ in $\mathrm{Top}(C \to D)$ is given. Then

$$[\check{\Sigma}_n{}^C X, Y]_D{}^X = [X, \hat{\Omega}_D{}^n Y]_Y{}^C$$

(c) Suppose $X \in \mathrm{Top}(C \to D)$ and $Y \in \mathrm{Top}(D = D)$. In the following equation take $C \to \Omega_D Y$ to be $\check{y}u$ and $X \to Y$ to be $\check{y}\hat{x}$. Then

$$[\check{\Sigma}_n{}^C X, Y]_D{}^X = [X, \Omega_D{}^n Y]_D{}^C$$

(6) Suppose a topological group G acts on C and D and $u: C \to D$ is a G-map. Then we have a category $\mathrm{Top}^G(C \to D)$ and the above constructions and equations are valid there also.

3. Relative Principal Fibrations

Let $Z \in \mathrm{Top}(D = D)$ and $P_D Z \to Z \in \mathrm{Top}(D = D)$ the canonical path-loop (so principal) fibration in the category. Let $X \in \mathrm{Top}(C \to D)$ and $f: X \to Z \in \mathrm{Top}(C \to D)$, where $Z \in \mathrm{Top}(C \to D)$ via $C \to D \to Z$. Consider the following pullback

$$\begin{array}{ccc} P = P_f & \longrightarrow & P_D Z \\ \downarrow{\scriptstyle p} & & \downarrow \\ X & \xrightarrow{\ f\ } & Z \end{array}$$

3.1 *Definition.* The map $P \to X$ described above is a *relative principal fibration* in $\mathrm{Top}(C \to D)$ (or, as in [5], a D-principal fibration).

Note that if f and X are in $\mathrm{Top}(D = D)$ then $P \to X$ is a principal fibration there and its properties are described in [4]. In particular if $D =$ point and f is a pointed map then $P \to X$ is a principal fibration in the usual sense and its properties are well known (see, for example, [Nomura, 7]).

We wish to study the lifting problem for $P \to X$. It is convenient to introduce one slight complication. Consider the following commutative diagram in $\mathrm{Top}(C \to D)$ and associated sequence.

$$\begin{array}{ccccc} & & P & \xrightarrow{\bar{f}} & P_D Z \\ & \overset{h}{\nearrow} & \downarrow p & & \downarrow \\ W & \xrightarrow{g} & X & \xrightarrow{f} & Z \\ & & \downarrow s & & \\ & & B & & \end{array} \tag{3.2}$$

$$[W, P]_B{}^C \xrightarrow{\beta} [W, X]_B{}^C \xrightarrow{\alpha} [W, Z]_D{}^C \tag{3.3}$$

Define $\beta = p_*$ and $\alpha = f_*$. $[W, Z]_D{}^C$ has a natural base point $\check{z}\hat{w}$.

3.4 THEOREM. (a) *The sequence* 3.3 *is exact.*
(b) *The group* $[W, \Omega_D Z]_D{}^C$ *acts on the set* $[W, P]_B{}^C$. *The action is defined below.*
(c) *The action in* (b) *is transitive on the stable set* $\beta^{-1}[g]$ *for any* $[g] \in [W, P]_B{}^C$.

Now fix g and h in the diagram 3.2. Consider

$$[W, \hat{\Omega}_B X]_X{}^C \xrightarrow{\delta} [W, \Omega_D Z]_D{}^C \tag{3.4}$$

Both sets are groups (see section 2) and δ, defined by $\delta[a] = [-H + (\hat{\Omega} f)a + H]$, $H = \bar{f}h$ is a homorphism.

3.4 THEOREM (cont.) (d) *The stability subgroup of the action on* $[h]$ *in* (c) *is Image* δ, *so* $\beta^{-1}[g]$ *is in one-to-one correspondence with the cokernel of* δ.

The action of (b) is defined as follows. First consider the map

$$(B \times {}_D\Omega_D Z) \times {}_B P \to P$$

defined by $(b, k, (x, m)) \to (x, k + m)$. Now $k, m\colon I \to Z$ and $k(1) = \check{z}\hat{b}(b)$ and $s(x) = b$ so $k(1) = \check{z}\hat{b}s(x) = \check{z}\hat{x}(x) = m(0)$ and hence the map is well defined. It is clearly continuous and a (C/B)-map. We deduce the following diagrams.

$$\begin{array}{c} [W, (B \times {}_D\Omega_D Z) \times {}_B P]_B{}^C \to [W, P]_B{}^C \\ \| \\ [W, B \times {}_D\Omega_D Z]_B{}^C \times [W, P]_B{}^C \\ \| \\ [W, \Omega_D Z]_D{}^C \times [W, P]_B{}^C \end{array}$$

It is easy to check that this defines an action of the group $[W, \Omega_D Z]_D^C$ on the set $[W, P]_B^C$.

3.5 *Note.* (a) If $P \to X$ is a principal fibration in Top$(D = D)$, $B = D$, *and* g, h *are the canonical zeros* $\hat{x}\hat{w}$, $\check{p}\hat{w}$ *respectively* then the above information is contained in [4].

(b) Suppose that $B = D =$ point, f is a pointed map, and g and h are arbitrary (making 3.2 commute). Then theorem 3.2 includes results of Nomura [6] and James-Thomas [2]. The other results of these authors (using restrictions on Z and X) can be generalized to the present context without difficulty (cf. note (a) of section 4).

(c) $D = K(\pi, 1)$, $Z = L_\psi(G, n)$ (see [5]). This case is useful for studying liftings in non-orientable fibrations.

(d) The definitions and results carry over to Top$^G(C \to D)$.

Proof of 3.4. (a) $\alpha[g] = 0$ is equivalent to the existence of a lifting of gf to $P_D Z$ and this, in turn, is equivalent to the existence of a lifting of g.

(b) It is easy to check that the map $(B \times_D \Omega_D Z) \times_B P \to P$ in Top$(C \to B)$ defines a homotopy action in the category [see Echman-Hilton, 1] and (b) follows immediately.

(c) Suppose $\beta[h] = \beta[h']$; that is, ph and ph' are $(C \mid B)$-homotopic. Then there is a $(C \mid B)$-map $h'': W \to P$ such that $ph'' = ph$ and a $(C \mid B)$-homotopy from h' to h''. This is because (1) $P \to X$ is a fibration over D since $P_D Z \to Z$ is and (2) (easy general fact) if $E \to Y$ is a fibration over D and the given map $Y \to D$ factors as $Y \to D' \to D$ then $E \to Y$ is a fibration over D'. Now define $d = d(h'', h): W \to \Omega_D Z$ by $d = \bar{f}h'' - \bar{f}h$. Then d is well defined because $p_1\bar{f}h'' = fph'' = fph = p_1\bar{f}h$. Also, d is a $(C \mid D)$-map and clearly $[d]\cdot[h] = [h''] = [h]$. This prove (c).

(d) We must prove that $u \cdot h \sim h$ if and only if $u \in Im\ \delta$. First suppose $u = \delta a = h' + \hat{\Omega}fa - h'$. Then $u \cdot h = u \cdot (g, h') = (g, h' + \hat{\Omega}fa - h' + h'): W \to P$. It is easy to check that $u \sim (g, h' + (\hat{\Omega}f)a) \sim (g, h')$ by a $(C \mid B)$-homotopy and so $u \cdot h \sim h$. The following lemma is helpful in proving the converse.

3.6 Lemma *Let* v, $v': W \to P_D Z$ *be maps in* $Top(C \to D)$. *Assume* $p_1 v = p_1 v'$: $W \to Z$ *and* $H: v \sim v'$ *is a homotopy in* $Top(C \to D)$. *Define* $u: W \to \hat{\Omega}_D Z$ *by* $u(w)(t) = H(w, t)\ (1)$, *Then* $v + u - v' \sim 0: W \to \Omega_D Z$ *by a homotopy in* $Top(C \to D)$. The lemma is not difficult to prove by standard techniques.

Now suppose $u \cdot h \sim h$ i.e. $(g, u + h') \sim (g, h')$ by a $(C \mid B)$-homotopy. The homotopy gives a self homotopy of g, i.e. a $(C \mid B)$-map $a: W \to \hat{\Omega}_B X$. The homotopy also gives $u + h \sim h$ and this homotopy covers $(\hat{\Omega}f)a$. The lemma gives $u + h + (\hat{\Omega}f)a - h \sim 0: W \to \Omega_D Z$ and hence $u \sim h - (\hat{\Omega}f)a - h$. So $u \sim \delta(-a): W \to \Omega_D Z$.

4. A Long Exact Sequence

Consider the diagram of section 3. *Assume that g and h are fixed.* In this section we extend 3.3 and 3.4 of section 3 to a long exact sequence.

4.1 Theorem. *The following is an exact sequence of pointed sets:*

$$\cdots [W, \hat{\Omega}_B{}^2X]_X \xrightarrow{\alpha_2} [W, \Omega_D{}^2Z]_D \xrightarrow{\gamma_1} [W, \hat{\Omega}_BP]_P \xrightarrow{\beta_1}$$

$$[W, \hat{\Omega}_BX]_X \xrightarrow{\alpha_1} [W, \Omega_DZ_D] \xrightarrow{\gamma_0} [W, P]_B \xrightarrow{\beta_0} [W, X]_B \xrightarrow{\alpha_0} [W, Z]_D\,.$$

The sequence consists of groups and homomorphisms after the third term and abelian groups after the sixth. (Here $[-, -]$ *means* $[-, -]^C$. *The zero elements of the sets and the maps are described below).*

Notes (a) Suppose $\check{s}:B \to X$ and $s\check{s} = 1$ and X is an H-space in $\mathrm{Top}(B = B)$ (in the obvious categorical sense). Do not suppose f is an H-map or even a map under B if $B = D$. Then it is not difficult to check the isomorphism:

$$[W, \hat{\Omega}_B{}^iX]_X{}^C \cong [W, \Omega_B{}^iX]_B{}^C$$

The replacement (with adjustment of maps) gives a simpler sequence. The first stage of a tower for $BO(n) \to BO$ (as in [5]) gives such a simplified sequence. There is a similar simplification in the more general case that $X \to B$ is itself a relative principal fibration.

(b) $Z = L_\psi(G, n), D = K(\pi, 1)$. The sequence is

$$\cdots [W, \hat{\Omega}_B{}^2X]_X{}^C \to H^{n-2}(W, C; G_\psi) \to [W, \hat{\Omega}_BP]_P{}^C \to [W, \hat{\Omega}_BX]_X{}^C$$

$$\to H^{n-1}(W, C; G_\psi) \to [W, P]_B{}^C \to [W, X]_B{}^C \to H^n(W, C; G_\psi).$$

$$\beta^{-1}[g] = \text{cokernel}\, [W, \hat{\Omega}_BX]_X \to H^{n-1}(W, C; G_\psi).$$

(c) $B = D, f \in \mathrm{Top}(D = D)$, and $g = \check{x}\hat{w}, h = \check{p}\hat{w}$, then this sequence reduces to the path-loop sequence of [4]. In particular, if $D = *$ and $g = h = *$ then this is the standard path-loop sequence (Nomura [7]).

(d) The proof given below is direct. If mild assumptions are made the sequence can be obtained by function space methods.

The zero elements are defined as follows: $D \to \Omega_D{}^iZ$ sends d to the constant loop at $\check{z}(d)$. $\Omega_D{}^iZ \to D$ is the evaluation map. Denote these maps by $\check{z}$ and $\hat{z}$ respectively (abuse of notation). The zero of $[W, \Omega_D{}^iZ]_D$ is $\check{z}\hat{w}$. $X \to \hat{\Omega}_B{}^iX$ sends x to the constant loop at x and $\hat{\Omega}_B{}^iX \to X$ is the evaluation. Denote these by $\check{x}$ and $\hat{x}$ respectively. $\check{p}$ and $\hat{p}$ are defined similarly. The zero of $[W, \hat{\Omega}_B{}^iX]_X$ is $\check{x}g$ and the zero of $[W, \hat{\Omega}_B{}^iP]_P$ is $\check{p}h$.

The homomorphisms are defined as follows.

(a) $\alpha_n:[W, \hat{\Omega}_B{}^nX]_X \to [W, \Omega_D{}^nZ]_D$. Define $\alpha_n[k] = [G_0]$ where $G: W \times I \to \hat{\Omega}_D{}^nZ$ covers $H = \bar{f}h$ and $G_1 = \hat{\Omega}^nfu$. It is not difficult to check that the result is independent of the choice of G and is a homomorphism. Also $\alpha_0 = f_*$ and $\alpha_1[k = [H + \hat{\Omega}fk - H]$.

(b) $\beta_n:[W, \hat{\Omega}_B{}^nP]_{\hat{P}} \to [W, \hat{\Omega}_B{}^nX]_X \qquad \beta_n = (\hat{\Omega}^np)_*$

(c) $\gamma_n:[W, \Omega_D{}^{n+1}Z]_D \to [W, \hat{\Omega}_B{}^nP]_P\ \gamma_n[k] = [k]\cdot[h_n]$ where h_n is defined from

$H = \bar{f}h$ and the action is defined by

$$\hat{\Omega}_B{}^n\nu : \hat{\Omega}_B{}^n((B \; X_D \; \Omega_D Z) \; X_B \; P) \to \hat{\Omega}_B{}^n P$$

and ν is the action of part 3.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
[W, \hat{\Omega}_B{}^n X]_X & \xrightarrow{\alpha_n} & [W, \Omega_D{}^n Z]_D & \xrightarrow{\gamma_{n-1}} & [W, \hat{\Omega}_B{}^{n-1} P]_P & \xrightarrow{\beta_{n-1}} & [W, \hat{\Omega}_B{}^{n-1} X]_X & \xrightarrow{\alpha_{n-1}} & [W, \Omega_D{}^{n-1} Z]_D \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
[\Sigma^{n-1} W, \hat{\Omega}_B X]_X & \longrightarrow & [\Sigma^{n-1} W, \Omega_D Z]_D & \longrightarrow & [\Sigma^{n-1} W, P]_B & \longrightarrow & [\Sigma^{n-1} W, X]_B & \longrightarrow & [\Sigma^{n-1} W, Z]_D
\end{array}$$

In the top row $[-, -]$ means $[-, -]^C$ and in the bottom row it means $[-, -]^W$. Σ means $\hat{\Sigma}^C$ everywhere. The vertical arrows arise from the observation of section 2 that (for example) $\hat{\Omega}_B{}^n X = \Omega_X{}^{n-1}\hat{\Omega}_B X$ and the adjoint relations. The bottom row almost fits into the scheme of section 3. If it fitted exactly we would be done. The problem is that the map $\bar{f}h : W \to P_D Z$ doesn't factor through D as we required for $C \to P_D Z$. However, the maps of section 3 can be modified slightly to take this into account and exactness of the 5 term sequence then follows.

UNIVERSITY OF KANSAS

REFERENCES

[1] ECKMAN AND P. HILTON, *Operators and cooperators in homotopy theory*, Math. Ann., **141** (1960), 1–21.

[2] I. M. JAMES AND E. THOMAS, *An approach to the enumeration problem for non-stable vector bundles*, J. Math. Mech., **14** (1965), 485–506.

[3] J. F. MCCLENDON, Thesis, Berkeley (1966) (Abstract published in Dissertation Abstracts, February 1967, p. 2785-B).

[4] ———, *Higher order twisted cohomology operations*, Invent. Math. **7** (1969), 183–214.

[5] ———, *Obstruction theory in fiver spaces*, Math. Z. **120** (1971), 1–17.

[6] Y. Nomura, On a lifting theorem, Reports from Liberal Arts and Science Faculty, Shizuoda University, vol. **3**, No. 2 (1961).

[7] ———, *On mapping sequences*, Nagoya Math. J. (1960), 111–45.