CODIMENSION ONE ANOSOV FLOWS

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Introduction

One of the main objectives in the Qualitative Theory of Dynamical Systems is the study of the orbit structure of diffeomorphisms and flows, and their classification under the equivalence relation given by topological conjugacy. Great progress has been made in the case of Anosov Systems (see [1] and [29] for definitions and background). It was proved originally by Anosov that Anosov Systems on compact manifolds are C^1-structurally stable (there are also proofs of this fact by Moser [21] and Mather [20]).

For codimension one Anosov diffeomorphisms, Franks [9] has given a practically complete solution to the classification problem. He proved that a codimension one Anosov diffeomorphism on a compact manifold M is topologically conjugate to a hyperbolic toral isomorphism, provided that the nonwandering set of the diffeomorphism is all of M. Furthermore, he proved that two codimension one Anosov diffeomorphisms are topologically conjugate if and only if they are π_1-conjugate. Later, it was proved by Newhouse [22] that a codimension one Anosov diffeomorphism on a compact manifold has as its nonwandering set the whole manifold.

For codimension one Anosov flows, one does not have such a complete classification theorem as the one given by Franks for diffeomorphisms. This paper gives some contributions in this line.

In this work, we obtain among others, the following results:

Theorem 1.1. *If $f_t: M \to M$ is a codimension one Anosov flow in the compact manifold M, then $\Omega(f_t) = M$. Here $\Omega(f_t)$ denotes the nonwandering set.*

Corollary 1.1 *The periodic orbits of f_t are dense in M and f_t is topologically transitive.*

Theorem 3.1. *The universal covering space of M is diffeomorphic to euclidean space.*

Theorem 3.3. *The center of $\pi_1(M)$ is either trivial or else it is free cyclic.*

Theorem 3.4. *If $\dim M = 3$ and if the center, L, of $\pi_1(M)$ is free cyclic and $\pi_1(M)/L$ is torsion-free then $\pi_1(M)/L$ is isomorphic to the fundamental group of a compact surface M^2, of genus greater than one. Furthermore, if $a \in H^2(M^2, \mathbf{Z})$ denotes the central extension*

$$a: 0 \to Z \to \pi_1(M) \to \pi_1(M^2) \to 0,$$

then M is diffeomorphic to the principal circle bundle over M^2 associated with a. Hence, M admits a principal S^1 action.

THEOREM 4.1. *$f_t: M \to M$ is topologically equivalent to the suspension of a hyperbolic toral isomorphism if and only if $H^1(M, \mathbf{R}) = \mathbf{R}$ and every periodic orbit represents a nontrivial element of $H_1(M, \mathbf{R})$.*

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§1. Preliminaries

In all that follows M will be a C^∞ manifold which is connected and without boundary. TM will denote the tangent bundle of M and T_xM the fibre at x. If $g: M \to M$ is a C^r map $(r \geq 1)$ we will denote by D_xg the derivative at x and by $Dg: TM \to TM$ the derivative bundle map induced by g.

DEFINITION 1.1. Let $f_t: M \to M$ be a C^r flow $(r \geq 1)$, generated by the vector field X; we say that f_t is an *Anosov flow* if:

(1.1) $$X(m) \neq 0 \ \forall \ m \in M.$$

(1.2) There exists a continuous splitting of TM into a Whitney direct sum, $TM = E^s \oplus E^u \oplus E^1$, which is invariant under Df_t and E^1 denotes the line bundle spanned by X.

(1.3) There exists constants $C, C^1, \lambda > 0$ such that for every $t > 0$
a) $\| Df_t(V) \| \geq C e^{\lambda t} \| V \|$ if $V \in E^u$
b) $\| Df_t(W) \| \leq C^1 e^{-\lambda t} \| W \|$ if $W \in E^s$
where $\| \cdot \|$ denotes the norm induced by a riemannian metric.

Examples of Anosov flows can be obtained by suspension of Anosov diffeomorphisms and by the geodesic flows of compact, connected, riemannian manifolds with negative curvature. The article [34] is an extensives study of Anosov flows on infra-homogeneous spaces. For further details, general information, references and terms not defined here we refer to the papers [1], [2], [12], [29].

Given a flow $F_t: M \to M$ we denote by $\Omega(F_t)$ its *nonwandering set.* Explicitely. $\Omega(F_t)$ is defined by $\Omega(F_t) = \{x \in M$: Given any neighborhood U of x and $T > 0$, there exists $t_0 \ X \ T$, such that $F_{t_0}(U) \cap U \neq \emptyset\}$. We also denote by $\Omega(X)$ the nonwandering set of the flow generated by the vector field X.

An Anosov flow $f_t: M \to M$ is said to be of *codimension 1* if either $\dim E^u = 1$ or $\dim E^s = 1$.

It has been conjectured that if $f_t: M \to M$ is an Anosov flow, then $\Omega(f_t) = M$. The following theorem asserts that such is the case if f_t is of codimension 1.

THEOREM 1.1. *If $f_t: M \to M$ is an Anosov flow of codimension 1 in the compact manifold M (this is always the case if $\dim M \leq 4$), then $\Omega(f_t) = M$.*

Anosov flows satisfy Axioms A' and B' of Smale (see [29] for the definition of these axioms). This is a consequence of the C^1-structural stability of Anosov flows and a general density theorem of Pugh [27]. Hence, we have the following:

COROLLARY 1.1. *If $f_t\colon M \to M$ is a codimension 1 Anosov flow then the periodic orbits of f_t are dense in M and f_t is topologically transitive.*

Before giving the proof of Theorem 1.1, we will need several results valid for any Anosov flow. The proof of Theorem 1.1 will be postponed to §2.

SPECTRAL DECOMPOSITION THEOREM (Smale [29]). *If $f_t\colon M \to M$ is an Anosov flow, then $\Omega(f_t)$ is the disjoint union of closed, invariant, indecomposable sets*:

$$\Omega(f_t) = \Omega_0 \cup \cdots \cup \Omega_m.$$

Furthermore, $f_t \mid \Omega_j$ is topologically transitive. The sets Ω_j, $(0 \leq j \leq m)$ are called *basic sets* (see also [23]).

DEFINITION 1.2. A *codimension k foliation of class C^s*, $s \geq 0$, of an n-dimensional manifold M, is a decomposition, $\mathfrak{F}$, of M into disjoint, connected subsets, called the leaves of the foliation, such that for each $m \in M$ there exists local C^s coordinates $(x_1, \cdots, x_n)$ so that in a neighborhood of m the leaves are described by the equations $x_1 =$ constant, $\cdots$, $x_k =$ constant.

The following is the Stable Manifold Theorem for codimension 1 Anosov flows (see [1], [15] and [16]).

PROPOSITION 1.1. *If $f_t\colon M \to M$ is a C^r Anosov flow on M with $r \geq 2$ and dim $E^u = 1$, then the distributions E^u and E^s are uniquely integrable with leaves of class C^r. More precisely, there exist foliations $\mathcal{U}$ and $\mathcal{S}$ called the strongly unstable and strongly stable foliations, respectively, so that*

a) Both $\mathcal{U}$ and $\mathcal{S}$ are invariant under the flow:

$$f_t(u(x)) = u(f_t(x))$$
$$f_t(s(x)) = s(f_t(x))$$

where, for $z \in M$, $u(z)$ and $s(z)$ denote the leaves through z of $\mathcal{U}$ and $\mathcal{S}$, respectively.

b) $u(x) = u(y)$ if and only if $\lim_{t\to\infty} d(f_{-t}(x), f_{-t}(y)) = 0$
$s(x) = s(y)$ if and only if $\lim_{t\to\infty} d(f_t(x), f_t(y)) = 0$

where the distance d is the one given by the riemannian metric on M.

c) Each leaf $u(x)$ is a C^r injectively immersed copy of $\mathbf{R}$ ($\mathbf{R}$ denotes the real line). Analogously, each leaf $s(x)$ is a C^r injectively immersed copy of $\mathbf{R}^{n-2}$. Furthermore, the foliations $\mathcal{U}$ and $\mathcal{S}$ are tangent to the continuous distributions E^u and E^s, respectively, and their leaves are C^r-close on compact sets.

It follows from the invariance of $\mathcal{U}$ and $\mathcal{S}$ under the flow that there is another pair of foliations $\hat{\mathcal{U}}$ and $\hat{\mathcal{S}}$, tangent to $E^u \oplus E^1$ and $E^s \oplus E^1$, respectively, and whose leaves $\hat{u}(x)$ are defined by

$$\hat{u}(x) = \bigcup_{t\in\mathbf{R}} \{u(f_t(x))\} = \bigcup_{y\in\gamma(x)} \{u(y)\}$$

$$\hat{s}(x) = \bigcup_{t\in\mathbf{R}} \{s(f_t(x))\} = \bigcup_{y\in\gamma(x)} \{s(y)\}$$

where $\gamma(x)$ denotes the orbit throught x. Clearly, we have that $f_t(\hat{s}(x)) = \hat{s}(x)$ and $f_t(\hat{u}(x)) = \hat{u}(x)$, for all $t \in \mathbf{R}$.

If $u \in \mathcal{U}$ and $x_1, x_2 \in u$, then from inequalities (1.3) it follows easily that for every $t > 0$

$$d(f_t(u); f_t(x_1), f_t(x_2)) \geq C e^{\lambda t} d(u; x_1, x_2) \tag{2.1}$$

$$d(f_{-t}(u); f_{-t}(x_1), f_{-t}(x_2)) \leq C^{-1} e^{-\lambda t} d(u; x_1, x_2). \tag{2.2}$$

Similarly, if $s \in \mathcal{S}$ and $x_1, x_2 \in s$, then

$$d(f_t(s); f_t(x_1), f_t(x_2)) \leq C' e^{-\lambda t} d(s; x_1, x_2) \tag{2.3}$$

$$d(f_{-t}(s); f_{-t}(x_1), f_{-t}(x_2)) \geq (C')^{-1} e^{\lambda t} d(s; x_1, x_2). \tag{2.4}$$

Where $d(u; \cdot, \cdot)$ and $d(s; \cdot, \cdot)$ denote the distances, induced on the leaves of $\mathcal{U}$ and $\mathcal{S}$, respectively, by the riemannian metric in M.

Using these inequalities, the contraction mapping theorem and the invariance of $\mathcal{U}$ and $\mathcal{S}$, the following is easy to prove:

PROPOSITION 1.2. *The leaves of $\hat{\mathcal{U}}$ (respectively $\hat{\mathcal{S}}$) are C^r immersed submanifolds diffeomorphic (with their intrinisc topology) to either $\mathbf{R}^2$ (respectively $\mathbf{R}^{n-1}$) or $\mathbf{R} \times S^1$ (respectively $\mathbf{R}^{n-2} \times S^1$). The latter cases hold if and only if the leaf contains exactly one periodic orbit. Furthermore, if the periodic orbit γ belongs to the leaf $\hat{u}$ (respectively $\hat{s}$) then the holonomy group of $\hat{u}$ (respectively $\hat{s}$) is isomorphic with $\mathbf{Z}$ and γ represents a generator. Evidently, γ is also a generartor of $\pi_1(\hat{u})$ (respectively $\pi_1(\hat{s})$). The germs representing nontrivial holonomy are local generic contractions or expansions.*

Remark 1.2. Hirsch and Pugh [15] have proved that in our case $\hat{\mathcal{S}}$ is a C^1 foliation, that is to say, $E^s \oplus E^1$ is a C^1 distribution. However, J. Plante [26] has examples where $\mathcal{S}$ is not C^1. Hirsch and Pugh require $r \geq 2$.

Let $f_t: M \to M$ be a codimension one Anosov flow generated by the vector field X. By the C^1-structural stability theorem there is no loss of generality if we assume that f_t is of class C^∞. Since the roles of E^u and E^s are reversed when we replace X by $-X$ we can assume also that dim $E^u = 1$. Again, there is no loss of generality if we assume that M is orientable and E^u is an orientable line bundle since we can achieve this by taking double coverings.

By Proposition 1.1, each $u \in \mathcal{U}$ is diffeomorphic to $\mathbf{R}$. We say that $y_1 < y_2$ for two distinct points of M, if y_1 and y_2 lie on the same leaf $u \in \mathcal{U}$ and if the oriented arc from y_1 to y_2 has the orientation of u. If $y_1 < y_2$ we let

$$[y_1, y_2] = \{y \in u: y_1 \leq y \leq y_2\}.$$

The length of $[y_1, y_2]$ will be denoted by $\ell[y_1, y_2]$. If $y \in u$ and $a > 0$ we will denote by $y + a$ (respectively $y - a$), the unique point of u such that $y + a > y$ (respectively $y - a < y$) and $\ell[y, y + a] = a$ (respectively $\ell[y - a, y] = a$). Analogously, we define the half-open intervals $[y_1, y_2)$, $(y_1, y_2]$, $(-\infty, y]$, $[y, \infty)$, $(-\infty, y)$, (y, ∞). For a subset $A \subset u$ we define sup A and inf A, in the obvious way.

The following proposition guarantees that we can take sufficiently big product neighborhood sets [29]. We let

$$B^n = \{(x_1, x_2, \cdots, x_n) = x \in \mathbf{R}^n : x_1^2 + \cdots + x_n^2 \le 1\}$$

PROPOSITION 1.3. (*i*) *Given $u \in \mathcal{U}$ and $y_1, y_2 \in u$ with $y_1 < y_2$, there exists a homeomorphism $\varphi: B^{n-1} \times B^1 \to V$, where V is a closed neighborhood of (y_1, y_2) such that*

a) *For every $t \in B^1$, $\varphi(B^{n-1} \times \{t\}) \subset \hat{s}\,(\varphi(0, t))$*
b) *For every $x \in B^{n-1}$, $\varphi(\{x\} \times B^1) \subset u\,(\varphi(x, 0))$*
c) $[y_1, y_2] = \varphi(\{0\} \times B^1)$.

(ii) *Given $\hat{s} \in \hat{\mathcal{S}}$, $y \in \hat{s}$ and a topological embedding $h: B^{n-1} \to \hat{s}$ such that $h(k_0) = y$ for some $k_0 \in B^{n-1}$: then there exists a homeomorphism $\psi: B^{n-1} \times B^1 \to W$ such that*

a′) $\psi(B^{n-1} \times \{0\}) = h(B^{n-1})$
b′) *For every $t \in B^1$, $\psi(B^{n-1} \times \{t\}) \subset \hat{s}\,(\psi(k_0, t))$*
c) *For every $k \in B^{n-1}$, $\psi(\{k\} \times B^1) \subset \hat{u}\,(\psi(k, 0))$.*

We will not prove this proposition. We only indicate that part (i) is proved in exactly the same way as the proof of the long flow box theorem given in [25]. For part (ii) we remark that h is right and left displaceable in the sense of Novikov (see §3 of [24],) because B^{n-1} is simply connected, and from this the existence of ψ follows directly. The sets V and W are called *product neighborhoods* relative to $[y_1, y_2]$ and $h(B^{n-1})$, respectively.

If K is a subset of M we let

$$u(K) = \bigcup_{x \in K} u(x),\ s(K) = \bigcup_{x \in K} s(x),\ \hat{u}(K) = \bigcup_{x \in K} \hat{u}(x), \text{ and } \hat{s}(K) = \bigcup_{x \in K} \hat{s}(x)$$

If K is invariant, then $u(K) = \hat{u}(K)$ and $s(K) = \hat{s}(K)$. In particular $\hat{s}(x) = \hat{s}(\gamma(x))$ and $\hat{u}(x) = \hat{u}(\gamma(x))$, where $\gamma(x)$ will always denote the orbit through x.

If $\Omega(f_t) = \Omega_0 \cup \cdots \cup \Omega_m$ is the spectral decomposition for $\Omega(f_t)$, then a basic set Ω_i for which $u(\Omega_i) = \Omega_i$ is called a *sink* and a basic set Ω_j for which $s(\Omega_j) = \Omega_j$ is called a *source*. It is a well-known fact that among the basic sets there is at least one which is a sink and at least one which is a source. It is easy to see that if Ω_i is a source then $u(\Omega_i)$ is open and that if Ω_j is a sink then $s(\Omega_j)$ is open. Therefore if we show that some source Ω_0 is also a sink then we would have $\Omega_0 = u(\Omega_0) = s(\Omega_0)$ and Ω_0 would be both open and closed. Since M is connected, Theorem 1.1 would follow. This is the key observation of Newhouse in [22]. The proof to be given in §2 is modeled after his. Considerable difficulties arise from the fact that in our case the foliations $\hat{\mathfrak{U}}$ and $\hat{\mathfrak{S}}$ have nontrivial holonomy.

§2. Proof of Theorem 1.1

Let Ω_0 be a source. We will show that Ω_0 must also be a sink. The key lemma is the following:

LEMMMA 2.1. (1) *If* $x \in \Omega_0$ *then* $(x, \infty) \cap \hat{s}(x) \neq \emptyset$.
(2) *If* $x \in \Omega_0$ *then* $(-\infty, x) \cap \hat{s}(x) \neq \emptyset$.

We will only prove part (1) since the proof of part (2) is similar. For the sake of simplicity we will divide the proof of Lemma 2.1 in various steps.

Proof of Lemma 2.1 *when* $\dim M > 3$. Let $A = \{x \in \Omega_0 : (x, \infty) \cap \hat{s}(x) = \emptyset\}$. We have to show that A is empty. Suppose $A \neq \emptyset$. We will arrive to a contradiction through a series of propositions.

PROPOSITION 2.1. *A is a closed invariant set.*

Proof. From Proposition 1.3 it is evident that A is closed, and it is obvious that A is invariant.

PROPOSITION 2.2. *A consists entierly of periodic orbits.*

Proof. Let $x \in A$. Let y be a point in the α-limit set of $\gamma(x)$. Let $V(y)$ be a product neighborhood around y. If $\gamma(x)$ is not compact then $\lim_{t\to\infty} d(\hat{s}(x); x, f_{-t}(x)) = \infty$ and therefore, if $\gamma(x)$ is not compact there exists $T_0 < 0$ such that $(f_{T_0}(x), \infty) \cap \hat{s}(x) \neq \emptyset$, contradicting the invariance of A. Therefore, $\gamma(x)$ is compact.

Let $x_0 \in A$ and $\gamma = \gamma(x_0)$ the periodic orbit through x_0. Since Ω_0 is a source $\hat{s}(x_0) \subset \Omega_0$. Since γ is the unique periodic orbit contained in $\hat{s}(x_0)$ we have that for every $x \in \hat{s}(x_0) - \gamma$, $(x, \infty) \cap \hat{s}(x_0) \neq \emptyset$. Let us set $\Gamma = \hat{s}(x_0) - \gamma = \hat{s}(\gamma) - \gamma$. We define the function $\varphi : \Gamma \to \mathbf{R}$ by $\varphi(x) = \inf\{\ell[x, y] : y \in (x, \infty) \cap \hat{s}(\gamma)\}$.

PROPOSITION 2.3. $\varphi(x) > 0$ *for all* $x \in \Gamma$.

Proof. If $\varphi(x) = 0$ for some $x \in \Gamma$, then it would follow that $\hat{s}(x_0)$ would self-accumulate and, using long product neighborhoods, one could prove that $(x_0, \infty) \cap \hat{s}(x_0) \neq \emptyset$ which would be a contradiction.

PROPOSITION 2.4. *For* $x \in \Gamma$ *let* $m(x)$ *be the point in* (x, ∞) *such that* $\ell[x, m(x)] = \varphi(x)$. *Then there exists a fixed* $y \in M$ *such that* $m(x) \in \hat{s}(y)$ *for all* $x \in \Gamma$. *In other words: all the points* $m(x)$ *lie in the same stable leaf when* x *varies in* Γ.

Proof. Since we are assuming that $\dim M > 3$, it follows by the Jordan-Brower separation theorem [32], that Γ is connected. If $z \in M$ and we let $V_z = \{x \in \Gamma : m(x) \in \hat{s}(z)\}$ then by Proposition 1.3, it follows easily that V_z is open in Γ. If $\hat{s}(z_1) \neq \hat{s}(z_2)$ then $V_{z_1} \cap V_{z_2} = \emptyset$. This finishes the proof since Γ is connected. Hence, we have a well-defined map $m : \Gamma \to \hat{s}(y)$. It is an obvious consequence of its definition that m is injective.

PROPOSITION 2.5. *Let y be as in Proposition 2.4. The function $m:\Gamma \to \hat{s}(y)$ has the following properties:*

a) m preserves the flow: $m(f_t(x)) = f_t(m(x))$

b) m is a homeomorphism onto its image

c) *$\hat{s}(y)$ contains a periodic orbit γ_1*

d) $m(\Gamma) = \hat{s}(\gamma_1) - \gamma_1$.

Proof. Parts a) and b) are obvious consequences of the definition of m and the theorem of invariance of domain (since m is injective).

Proof of c). Given a periodic orbit $\bar{\gamma}$, of f_t, a subset $F \subset \hat{s}(\bar{\gamma})$ is called a "*fence*" for $\bar{\gamma}$ if F is a topological submanifold of $\hat{s}(\bar{\gamma})$, homeomorphic to $s^{n-3} \times S^1$, and such that every orbit in $\hat{s}(\bar{\gamma}) - \bar{\gamma}$ intersects F in exactly one point. Fences associated to periodic orbits always exist and, in fact, they can be taken as boundary components of fundamental domains. Let F be a fence for γ and let us assume that $\hat{s}(y)$ does not contain a periodic orbit; then it is easy to see that there exists a point $y_0 \in \hat{s}(y)$ such that $m(F) \cap s(y_0) = \emptyset$. By b) it follows that $m(F)$ is a topological submanifold of $\hat{s}(y)$ homeomorphic to $S^{n-3} \times S^1$, and by a), an orbit in $\hat{s}(y)$ which intersects $m(F)$ has exactly one point of intersection. Then one can define a continuous and injective map $\delta:m(F) \to s(y_0)$ given by $\delta(p) = \gamma(p) \cap s(y_0)$. The compactness of $m(F)$ and the theorem of the invariance of domain lead to a contradiction since $s(y_0)$ is, by Proposition 1.1, homeomorphic to $\mathbf{R}^{n-2}$. Therefore $\hat{s}(y)$ must contain a periodic obit γ_1. Obviously we have $m(F) \cap \gamma_1 = \emptyset$.

Proof of d). Let $\bar{F}$ be a fence for γ_1. The proof of d) reduces to show that $m(\Gamma)$ contains $\bar{F}$. Let ut define a map $\eta:m(F) \to \bar{F}$ by $\eta(p) = \gamma(p) \cap \bar{F}$. The map η is injective and continuous and the theorem of invariance of domain implies that $\eta(m(F)) = \bar{F}$. Using a) we conclude that $\bar{F} \subset m(\Gamma)$. This finishes the proof of Proposition 2.5.

Let us set $H = \bigcup_{x\in\Gamma} [x, m(x)]$ and let us fix $x_1 \in \Gamma$. For each $z \in [x_1, m(x_1)]$ let H_z denote the connected component of $\hat{s}(z) \cap H$ which contains z. The rest of the steps for the completion of Lemma 2.1 (still under the assumption that $\dim M > 3$) aim at proving that for each $z \in [x_1, m(x_1)]$ there exists a single orbit $\gamma_z \in \hat{s}(z)$ such that $H_z = \hat{s}(z) - \gamma_z$ and that, furthermore,

$$\bigcup_{z\in[x_1,\, m(\gamma_1)]}\gamma_z \subset \hat{u}(\gamma).$$

Hence, in particular, $\gamma_1 \subset \hat{u}(\gamma)$, which is an absurdity. We observe that, under the assumption that $A \neq \emptyset$, γ_1 must be different from γ. This is a consequence of the fact that a periodic orbit must have a homoclinic point and therefore if γ were equal to γ_1 then, for all $x \in \gamma$, $(-\infty, x) \cap \hat{s}(\gamma) \neq \emptyset$, and one can check that this is in contradiction to the way that the function φ, in Proposition 2.2, was chosen.

PROPOSITION 2.6. *For each $z \in [x_1, m(x_1)]$ the map $\pi_z:H_z \to \Gamma$ defined by $\pi_z(w)$*

$= x$ *if* $w \in [x, m(x)]$, *is a covering map. The flow restricted to* Γ *lifts, under* π_z, *to the flow restricted to* H_z. *Clearly* $\pi_z^{-1}(\Gamma)$ *is an open invariant subset of* H_z.

Proof. It is easy to see that π_z is surjective. Every point in Γ is evenly covered. To see this it is enough to consider for each $x \in \Gamma$ a product neighborhood relative to the segment $[x, m(x)]$. The other assertions are self-evident.

PROPOSITION 2.7. *For each* $z \in [x_1, m(x_1)]$, *the set* $\hat{s}(z) - H_z$ *is a non-empty connected, closed and invariant subset of* $\hat{s}(z)$.

Proof. The invariance of $\hat{s}(z) - H_z$ and the fact that H_z is open in $\hat{s}(z)$ follows implicitly from Proposition 2.6. Therefore it is only left to us to prove that $\hat{s}(z) - H_z$ is connected and non-empty. First, let us suppose that $\hat{s}(z)$ does not contain a periodic orbit. In this case $\hat{s}(z)$ is diffeomorphic to $\mathbf{R}^{n-1}$. Let F be a fence associated to γ. It is easy to see that $\pi_z^{-1}(F)$ is connected. Let $\tilde{F}_z = \pi_z^{-1}(F)$ and let us define the injective and continuous map $\psi_z:\tilde{F}_z \to s(z)$ by $\psi_z(y) = s(z) \cap \gamma(y)$. Here we are using the fact that every orbit in $s(z)$ meets $\tilde{F}_x$ in at most one point, and meets $s(z)$ in exactly one point. Again by the theorem of invariance of domain it follows that ψ_z is a homeomorphism onto its image. The set $\tilde{F}_z$ is a covering space of F homeomorphic to $S^{n-3} \times \mathbf{R}$. Let $\mu: S^{n-3} \times \mathbf{R} \to \tilde{F}_z$ be a homeomorphism. For each real number t, $\psi_z(\mu(S^{n-3} \times \{t\}))$ is a homeomorph of an $(n-3)$-sphere in $s(z)$, and therefore is the boundary of a compact, connected set which we will denote by A_t^z. We can assume that if $t_1 \geq t_2$ then $A_{t_1}^z \subset A_{t_2}^z$. Therefore, the sequence $\{A_n^z\}$, $n = 1, 2, \cdots$, is a nested sequence of compact and connected sets. Hence $W_1 = \bigcap_{n=1}^{\infty} A_n^z$ is non-empty, compact and connected. On the other hand, using the one-point compactification of $s(z)$ one sees that $W_{-1} = s(z) - \bigcup_{n=-1}^{-\infty} A_n^z$ is either empty or else is a nonempty connected set.

Claim. W_{-1} is empty. Let us suppose the contrary and let $x \in W_{-1}$, $y \in W_1$. Then $\lim_{t\to\infty} d(\hat{s}(z); f_t(x), f_t(y)) = 0$. Since $\hat{s}(z)$ is simply connected, and $\tilde{F}_z$ is a connected submanifold of codimension one in $\hat{s}(z)$, we have that $\hat{s}(z) - \tilde{F}_z$ is the disjoint union of two open sets V_0 and V_1, and we can assume that $x \in V_0$ and $y \in V_1$. From the compactness of F one can verify that $\tilde{F}_z$ has a uniform tubular neighborhood T_z totally contained in H_z. Since $\lim_{t\to\infty} d(\hat{s}(z); f_t(x), f_t(y)) = 0$, and $f_t(x) \in V_0, f_t(y) \in V_1$ for all t, there exists $t_0 > 0$ such that $f_{t_0}(x)$ and $f_{t_0}(y)$ belong to H_z. This implies that $x, y \in \psi_z(\mu(S^{n-3} \times \mathbf{R}))$ which is absurd. Hence W_{-1} must be empty.

If we set $W_z = \{\gamma(y): y \in W_1\}$ then W_z is also connected and $\hat{s}(z) - H_z = W_z$. This finishes the proof when $\hat{s}(z)$ does not contain a periodic orbit. If $\hat{s}(z)$ does contain a periodic orbit then by considering the universal covering of $\hat{s}(z)$ and lifting the flow, the metric, and the sets H_z, $\tilde{F}_z$, and proceeding as before, we prove that $\hat{s}(z) - H_z$ is connected and non-empty in all cases.

PROPOSITION 2.8. *For each* $z \in [x_1, m(x_1)]$, *the set* $\hat{s}(z) - H_z$ *consists of a single orbit* γ_z.

Proof. By Proposition 2.7 the set $\hat{s}(z) - H_z$ consists of either a single orbit or an uncountable number of orbits none of which is isolated. Let $A_1 = \{z \in [x_1, m(x_1)]: \hat{s}(z) - H_z$ consists of a single orbit$\}$ and $A_2 = \{z \in [x_1, m(x_1)]: \hat{s}(z) - H_z$ consists of an infinite number of orbits$\}$. Since $x_1 \in A_1$, $A_1 \neq \emptyset$. Obviously $A_1 \cap A_2 = \emptyset$. Using product neighborhoods one sees that A_1 is open and if $A_2 \neq \emptyset$ it also would be open. Since $[x_1, m(x_1)]$ is connected, the proposition follows.

PROPOSITION 2.9. $\bigcup_{z \in [x_1, m(x_1)]} \gamma_z \subset \hat{u}(\gamma)$.

Proof. For each $y \in M$ let $C_y = \{z \in [x_1, m(x_1)]: \gamma_z \subset \hat{u}(y)\}$. Let $z \in C_y$ and $y_0 \in \gamma_z$. Let K be a disc that contains y_0 and z in its interior. Let V be a product neighborhood relative to K. Then it becomes clear that for every $\bar{z} \in [x_1, m(x_1)] \cap V$, we must have $\gamma_{\bar{z}} = \gamma(p)$ with $p \in \hat{s}(\bar{z}) \cap \hat{u}(y)$, therefore $\bar{z} \in C_y$. Hence, C_y is open in $[x_1, m(x_1)]$. If $\hat{u}(y_1) \neq \hat{u}(y_2)$ then $C_{y_1} \cap C_{y_2} = \emptyset$. Since $[x_1, m(x_1)]$ is conncected it follows the proposition, because $\gamma_{x_1} = \gamma \subset \hat{u}(\gamma)$.

The assumption that $A \neq \emptyset$ leads, by Proposition 2.9 to the absurd conclusion that $\hat{u}(\gamma)$ contains two distinct periodic orbits. Hence $A = \emptyset$ and we have completed the proof of Lemma 2.1 when $\dim M > 3$.

Proof of Lemma 2.1 *when* $\dim M = 3$. Let A be defined as above. Then if $A \neq \emptyset$ again A is a closed invariant set that consists of periodic orbits. Let $x_0 \in A$. Then $\hat{s}(x_0)$ is an embedded copy of $S^1 \times \mathbf{R}$ and the periodic orbit $\gamma(x_0)$ separates $\hat{s}(x_0)$ into two connected components W_1 and W_2 each of which is an embedded copy of $S^1 \times \mathbf{R}$.

For every $x \in W_1$ we define the functions φ and m as above. We can do this since W_1 is connected and so there exists a point $y \in M$ such that $m(x) \in \hat{s}(y)$ for all $x \in W_1$.

The map $m: W_1 \to \hat{s}(y)$ is a homeomorphism onto its image, and this map preserves the flow. Let F be a simple closed curve, contained in W_1 that is transversal to the flow; then $m(F)$ is a simple closed curve in $\hat{s}(y)$ such that if it intersects an orbit then it has a unique point of intersection and $m(F)$ is collared by the flow.

PROPOSITION 2.10. *There exists a periodic orbit* $\bar{\gamma} \subset \hat{s}(y)$. *Furthermore,* $m(W_1)$ *is equal to one of the connected components in which* $\bar{\gamma}$ *divides* $\hat{s}(y)$.

Proof. If $\hat{s}(y)$ does not contain a periodic orbit, then $m(F)$ is the boundary of a 2-disc, since, $\hat{s}(y)$ with its intrinsic topology is diffeomorphic to $\mathbf{R}^2$. The index of the vector field (we restrict ourselves to the flow in $s(y)$) with respect to $m(F)$ is different from zero, since, under the circumstances, we can homotop $m(F)$ to a differentiable simple closed curve transversal to the flow (a topological manifold which is collared by a differentiable flow can be isotoped to a differentiable manifold transversal to the flow. See [37]). But this contradicts the Poincaré Index Theorem since there are no singularities. Hence, $\hat{s}(y)$ must contain a periodic orbit which we call $\bar{\gamma}$.

Now, let $\bar{W}_1$ and $\bar{W}_2$ be the two connected components in which $\bar{\gamma}$ divides

$\hat{s}(y)$ and let us assume that $m(W_1) \subset \bar{W}_1$. Since $\bar{\gamma}$ is a generic, attracting, periodic orbit in $\hat{s}(y)$, there exists a differentiable simple closed curve α, contained in $\bar{W}_1$, such that every orbit in $\bar{W}_1$ intersects α in exactly one point. Let $\delta:\alpha \to m(F)$ be the map defined by $\delta(x) = \gamma(x) \cap m(F)$. Then δ is continuous and injective and we must have $\delta(\alpha) = m(F)$. Therefore $\bar{W}_1 = m(W_1)$ and the proposition has been proven.

Let $H = \bigcup_{x \in W_1} [x, m(x)]$. Let us fix $\bar{x}_0 \in W_1$. For each $z \in [\bar{x}_0, m(\bar{x}_0)]$ let H_z be the connected component of $H \cap \hat{s}(z)$ that contains z. For each $z \in [\bar{x}_0, m(\bar{x}_0)]$ let $\pi_z: H_z \to W_1$ be the map defined by $\pi_z(y) = x$ if $y \in [x, m(x)]$. Then the proof given in Proposition 2.6 works as well to prove the following:

PROPOSITION 2.11. *The map $\pi_z: H_z \to W_1$ is a covering projection for each $z \in [\bar{x}_0, m(\bar{x}_0)]$. The flow lifts, under π_z, to the flow restricted to H_z, and $\pi_z^{-1}(W_1)$ is an open and invariant subset of H_z.*

Since $\dim M = 3$, it follows that for each $y \in M$, any orbit in $\hat{s}(y)$ separates $\hat{s}(y)$ in exactly two connected components.

PROPOSITION 2.12. *For each $z \in [\bar{x}_0, m(\bar{x}_0)]$ there exists an orbit $\gamma_z \subset \hat{s}(z)$ such that H_z is one of the connected components in which γ_z divides $\hat{s}(z)$.*

Proof. Let $F \subset W_1$ be a simple closed curve transversal to the flow in W_1. One proves easily that $\pi_z^{-1}(F)$ is connected. There are two cases

a) $\pi_z^{-1}(F)$ lifts to a simple closed curve in H_z. In this case, proceeding as in Proposition 2.10, one proves that there exists a periodic orbit γ_z in $\hat{s}(z)$ such that H_z is one of the connected components in which γ_z divides $\hat{s}(z)$. For such a z we have proved the proposition.

b) H_z does not contain a periodic orbit. In this case $\pi_z^{-1}(F)$ is homeomorphic to $\mathbf{R}$ and this curve is collared by the flow in $\hat{s}(z)$. Let $g_z: \pi_z^{-1}(F) \to s(z)$ be defined by

$$g_z(y) = \gamma(y) \cap s(z).$$

Then g_z is continuous and injective and $g_z(\pi_z^{-1}(F))$ is an open interval in $s(z)$ (here we think of $s(z)$, with its intrinsic topology as being homeomorphic to the real line). Using the fact that F is compact one shows that $\pi_z^{-1}(F)$ has a uniform tubular neighborhood totally contained in H_z.

We claim that $s(z) - g_z(\pi_z^{-1}(F))$ is connected. If this were not the case, then one can pick $z_0, z_1 \in s(z) - g_z(\pi_z^{-1}(F))$ lying in different components. Since $\pi_z^{-1}(F)$ has a uniform tubular neighborhood, there would exist $T > 0$ such that $f_T(z_0)$ and $f_T(z_1)$ are contained in H_z, which is a contradiction. Therefore either $g_z(\pi^{-1}(F)) = s(z)$ or else there exists a point $\bar{z} \in s(z)$ such that $g_z(\pi^{-1}(F))$ is one of the components of $s(z) - \{\bar{z}\}$. Thus to prove the proposition we only need to prove the following:

Claim. $s(z) - g_z(\pi_z^{-1}(F)) \neq \emptyset$. To prove this claim let us consider the set $N = \bigcup_{x \in F} [x, m(x)]$. This set is homeomorphic to $S^1 \times [0, 1]$ and it is filled with the oriented and regular family of curves $\pi_y^{-1}(F)$, for $y \in [\bar{x}_0, m(\bar{x}_0)]$. This family determines a continuous flow on N without fixed points. Since each curve $\pi_y^{-1}(F)$ obviously admits a transversal segment, we can apply Poincaré-Bendixon theorem to conclude that the α-limit sets and ω-limit sets of such curves are periodic orbits of this continuous flow. Therefore, there exists a sequence of points $\{y_i\}$ contained in $\pi_z^{-1}(F)$, converging in M to a point $z_0 \in N$, and a periodic orbit γ_{z_0} contained in $\hat{s}(z_0)$ such that H_{z_0} is one of the connected components of $\hat{s}(z_0) - \gamma_{z_0}$. Let $\bar{y} \in \gamma_{z_0}$ and K a closed disc in $\hat{s}(z_0)$ that contains $\bar{y}$ and z_0 in its interior. Let V be a stable product neighborhood relative to K. Let n be large enough so that $y_n \in V$. Then one sees that $\gamma(p) \cap s(z) \in s(z) - g_z(\pi_z^{-1}(F))$ for some $p \in V \cap \hat{s}(y_n) \cap \hat{u}(\bar{y})$. This proves the claim.

For each $z \in [\bar{x}_0, m(\bar{x}_0)]$ let γ_z be as in Proposition 2.12. Using exactly the same arguments as in Proposition 2.9, one proves:

PROPOSITION 2.13. $\bigcup_{z \in [\bar{x}_0, m(\bar{x}_0)]} \gamma_z \subset \hat{u}(\gamma(x_0))$.

Thus if $A \neq \emptyset$ we arrive to the absurd conclusion that $\hat{u}(x_0)$ contains two distinct periodic orbits $\gamma(x_0)$ and $\bar{\gamma}$. Therefore $A = \emptyset$ and we have proved Lemma 2.1 for every dimension of M.

Hence, for every $x \in \Omega_0$ we have

$$(x, \infty) \cap \Omega_0 \neq \emptyset$$

and

$$(-\infty, x) \cap \Omega_0 \neq \emptyset.$$

From the fact that Ω_0 is compact we conclude immediately the following

LEMMA 2.2. *There exists $\beta > 0$ such that if $y_1, y_2 \in u(\Omega_0)$ with $y_1 < y_2$ and if $\ell[y_1, y_2] > \beta$; then $[y_1, y_2] \cap \Omega_0 \neq \emptyset$.*

Now we are able to prove Theorem 1.1.

Proof of Theorem 1.1. Let Ω_0 be a source and $x \in \Omega_0$ arbitrary. Let $y_1, y_2 \in u(x)$ and $y_1 < y_2$. By the expanding property of the foliation, there exists $T > 0$ such that $\ell[f_T(y_1), f_T(y_2)] > \beta$. Thus by Lemma 2.2 $[f_T(y_1), f_T(y_2)] \cap \Omega_0 \neq \emptyset$. Therefore $[y_1, y_2] \cap \Omega_0 \neq \emptyset$. Since this happens for every pair $y_1, y_2 \in u(x)$, with $y_1 < y_2$, it follows that $u(x) \cap \Omega_0$ is dense in $u(x)$. Since Ω_0 is closed it follows that $u(x) \subset \Omega_0$. Since x was arbitrary $u(\Omega_0) = \Omega_0$. Therefore Ω_0 is also a sink. Thus $\Omega(f_t) = M$. The theorem is proved.

§3. The Universal covering of a manifold supporting a codimension one Anosov flow

In his paper on foliations [24] Novikov proves that a compact manifold M admitting a codimension one Anosov flow has trivial second homotopy group (therefore when $\dim M = 3$ the universal covering of M is contractible). In this chapter we prove a sharper result proving that in fact, the universal covering of such a manifold is euclidean space. In particular, M is aspherical. From this result we derive some consequences related to the fundamental group of M. The center of the fundamental group is either trivial or else free cyclic. When dim $M = 3$ and the center, L, is non-trivial, then M is diffeomorphic to a principal circle bundle over a compact 2-dimensional manifold V, of genus greater than one, provided that $\pi_1(M)/L$ is torsion-free. This circle bundle is classified by $a \in H^2(V, \mathbf{Z})$, corresponding to the central extension $0 \to \mathbf{Z} \to \pi_1(M) \to \pi_1(V) \to 0$.

First we will introduce some notations and prove several facts related to the associated foliations in a codimension one Anosov flow. From now on $f_t: M \to M$ will denote a smooth codimension on Anosov flow defined in the compact n-dimensional manifold M. We will always assume that dim $E^u = 1$. We will provide the universal covering of M with the complete riemannian metric which is the lifting of a fixed riemannian metric in M. We will always assume that M is oriented and that E^u is an oriented line bundle. If we lift f_t to the universal covering of M then this lifted flow is Anosov with respect to the lifted metric and we introduce an ordering in its one dimensional strongly unstable leaves as in chapter one, and use similar notations.

The following lemma about the C^1 codimension one foliation $\mathcal{S}$ can be proved by the methods of Lemma (5.1) of [9] (see also [13]).

Lemma 3.1. *Let $j: S^1 \to M$ be a smooth immesrion of the circle into M, which is transversal to the leaves of $\mathcal{S}$. Then j represents a nontrivial element of $\pi_1(M)$.*

Lemma 3.2. *For each leaf $s \in \mathcal{S}$ the inclusion map $i: s \to M$. induces a monomorphism $i_*: \pi_1(s) \to \pi_1(M)$.*

Proof. Clearly, we only need to prove this lemma when s contains a periodic orbit γ. In such a case γ represents a generator of $\pi_1(s)$ and therefore we only need to show any non-zero multiple of γ is not homotopic to a constant in M. It is easy to see that γ is freely homotopic to a smooth curve $\bar{\gamma} \subset \bar{u}(\gamma)$ which is transversal to $\mathcal{S}$. Hence by Lemma 3.1 no non-zero multiple of γ can be homotopic to a constant.

In all that follows $\tilde{M}$ will denote the universal covering of M, with covering projection $p: \tilde{M} \to M$. Let us denote by $\mathcal{W}$ the foliation in $\tilde{M}$ which is the lifting of the stable foliation $\mathcal{S}$ and by $w(x)$ the leaf through a point $\tilde{x} \in \tilde{M}$. The one dimensional foliation in $\tilde{M}$ which is the lifting of $\mathcal{U}$ will be denoted by $\tilde{\mathcal{U}}$ and the leaf through $\tilde{x}$ by $\tilde{u}(\tilde{x})$. Let us identify $\pi_1(M)$ with the group of covering (deck) transformations of the covering $p: \tilde{M} \to M$. Then each $\alpha \in \pi_1(M)$, thought of as a diffeomorphism $\alpha: \tilde{M} \to \tilde{M}$, preserves the leaves of $\mathcal{W}$ and $\tilde{\mathcal{U}}$. Furthermore, for each $\tilde{x} \in \tilde{M}$, $\alpha: \tilde{u}(\tilde{x}) \to \tilde{u}(\alpha(\tilde{x}))$ and $\alpha: w(\tilde{x}) \to w(\alpha(\tilde{x}))$ are orientation-preserving isometries.

As a corollary of the previous lemma we have:

Corollary 3.1. *Each leaf $w \in \mathcal{W}$ is a properly embedded copy of $\mathbf{R}^{n-1}$.*

Remark 3.1. If $h: \mathbf{R} \to \tilde{M}$ is a smooth embedding of the real line which is transversal to $\mathcal{W}$ then $h(t_1)$ and $h(t_2)$ lie in different leaves of $\mathcal{W}$ if $t_1 \neq t_2$. Otherwise, there would exist a simple closed curve transversal to $\mathcal{W}$ and homotopic to a constant. One gets a contradiction if one applies the arguments of Lemma 5.1 in [9]. Therefore, each $w \in \mathcal{W}$ does not self-accumulate.

Lemma 3.3. *If $\hat{s} \in \hat{\mathcal{S}}$ then $\hat{s}$ is dense in M.*

Proof. Let D denote the closure of $\hat{s}$. It suffices to show that D is open in M. Let $C = \{x \in M : u(x) \cap \hat{s} \neq \emptyset\}$. Then, C is open in M. Let s prove that $C = D$. If $x \in D$, let V be a product neighbrohood of x. Then $\hat{s} \cap V \neq \emptyset$ and $u(x) \cap \hat{s} \cap V \neq \emptyset$. Hence $x \in C$. On the other hand, let $x \in C$ be a point belonging to a periodic orbit of period $T > 0$ and let $y \in u(x) \cap \hat{s}$. Since $\lim_{n \to \infty} f_{-nT}(y) = x$, it follows that $x \in D$. By Corollary 1.1 the periodic orbits are dense. Therefore $C \subset D$. So we have that $D = C$.

Remark 3.2. Analogously, each $\hat{u} \in \hat{\mathcal{U}}$ is dense in M. The following is obvious:

Lemma 3.4. For each $\tilde{x} \in M$, $\bigcup_{\alpha \in \pi_1(M)} \alpha(w(\tilde{x}))$ is dense in $\tilde{M}$.

Remark 3.3. We observe that the leaves of the strongly stable or strongly unstable foliations may not be dense in M. Such is the case if the flow is a suspension.

Let $\tilde{u} \in \tilde{\mathcal{U}}$ and $\tilde{x}_1, \tilde{x}_2 \in \tilde{u}$ with $\tilde{x}_1 < \tilde{x}_2$. Let

$$A(\tilde{x}_1, \tilde{x}_2) = \bigcup_{\tilde{x}_1 < \tilde{z} < \tilde{x}_2} w(\tilde{z}).$$

Corollary 3.2. *The set $\{\alpha(A(\tilde{x}_1, \tilde{x}_2)) : \alpha \in \pi_1(M)\}$ is an open covering of $\tilde{M}$.*

The following is a description of the holonomy of $\hat{\mathcal{S}}$:

Proposition 3.1. *Let $w \in \mathcal{W}$ be such that there exists a nontrivial $\alpha \in \pi_1(M)$ such that $\alpha(w) = w$ then:*

i) *w is the lifting of $\hat{s} \in \hat{\mathcal{S}}$ where $\hat{s}$ contains a periodic orbit γ. Let $T > 0$ be the minimal period of γ.*

ii) *The subgroup of $\pi_1(M)$ that leaves w fixed, which we will denote by $G(w)$, is free cyclic. Let $\tilde{x} \in w$ be such that $p(\tilde{x}) \in \gamma$. Then if β is an appropriate generator of $G(w)$, we have $\beta^n(\tilde{x}) = \tilde{f}_{nT}(\tilde{x})$ for all n.*

iii) *If $\tilde{x} \in w$ and $p(\tilde{x}) \in \gamma$ then for every $\tilde{y} \in \tilde{u}(\tilde{x})$ and every integer n,*

$$\beta^n(w(\tilde{y})) \cap \tilde{u}(\tilde{x}) \neq \emptyset.$$

iv) *Each $\beta^n \in G(w)$ determines a C^1-diffeomorphism $h_{\beta^n} \tilde{u}(\tilde{x}) \to \tilde{u}(\tilde{x})$ defined by $h_{\beta^n}(y) = \beta^n(w(y)) \cap \tilde{u}(\tilde{x})$.*

This map has $\tilde{x}$ as its unique fixed point and it is generic. The correspondence $\beta^n \to h_{\beta^n}$ sets an isomorphism between $G(w)$ and the free cyclic subgroup of $\text{Diff}^1(\tilde{u}(\tilde{x}))$, generated by h_β. If $\mathcal{W}$ is of class C^r with $r \geq 1$ then h_β is also of class C^r.

In all that follows d will denote both the distance given by the fixed riemannian metric in M and the distance in $\tilde{M}$ which is obtained by lifting the riemannian metric in M to $\tilde{M}$.

We will denote by $\overline{\mathcal{W}}$ and $\overline{\mathcal{U}}$ the foliations which are the lifting of $\mathcal{S}$ and $\hat{\mathcal{U}}$

respectively. Their leaves through a point $x \in \tilde{M}$ will be denoted by $\bar{w}(x)$ and $\bar{u}(x)$, respectively. It follows from Corollary 3.1 and Remark 3.1 that for each $x \in \tilde{M}$, $\bar{w}(x)$ and $\bar{u}(x)$ are properly embedded copies of $\mathbf{R}^{n-2}$ and $\mathbf{R}^2$, respectively. Given two points $x, y \in \bar{w} \in \bar{\mathcal{W}}$, $d(\bar{w}; x, y)$ will denote the infimum of the lengths of piecewise smooth paths which lie in $\bar{w}$ and join x and y. By hyperbolicity we have $\lim_{t\to\infty} d(\bar{w}; \tilde{f}_t(x), \tilde{f}_t(y)) = \lim_{t\to\infty} d(\tilde{f}_t(x), \tilde{f}_t(y)) = 0$ for all $\bar{w} \in \bar{\mathcal{W}}$ and all $x, y \in \bar{w}$.

Proposition 3.2. *Given $N > 0$, there exists $T > 0$ such that $d(\tilde{f}_t(x), x) > N$ for all t such that $|t| > T$; and all $x \in \tilde{M}$.*

Proof. It suffices to show that given $N > 0$, there exists $T > 0$ such that $d(\tilde{f}_t(x), x) > N$ for all $t > T$ and all $x \in \tilde{M}$. Let us assume the contrary. Let $K \subset \tilde{M}$ be a compact fundamental domain (i.e. a compact set K such that $p(K) = M$). Then, under the assumption, there exists a sequence $t_n \to \infty$ and a sequence $\{x_n\}$ in K such that $x_n \to y$ for some $y \in K$ and $d(\tilde{f}_{t_n}(x_n), x_n) \leq N$. Let C be a product neighborhood which contains y in its interior. We can assume that the sequence $\{x_n\}$ is contained in C. Then, $\tilde{f}_{t_n}(C) \cap \bar{K} \neq \emptyset$, where $\bar{K} = \{x \in \tilde{M}: d(x, K) \leq N\}$. But one can see immediately that this implies that $\bar{u}(y)$ self-accumulates which is a contradiction since $\bar{u}(y)$ is a properly embedded copy of $\mathbf{R}^2$.

Remark 3.4. From Proposition 3.2 it is very easy to prove that given any compact set $K \subset \tilde{M}$ and $N > 0$, there exists $T > 0$ such that $K \cap \tilde{f}_t(\bar{w}(K)) = \emptyset$ for all t with $|t| > T$, where $\bar{w}(K) = \bigcup_{x\in K} \bar{w}(x)$.

Proposition 3.3. *If $w \in \mathcal{W}$ is such that there exists $\alpha \in \pi_1(M)$ with $\alpha(w) = w$, then, given $N > 0$ arbitrary, there exists an integer $m > 0$ such that $d(\bar{w}, \alpha^n(\bar{w})) > N$ for all $\bar{w} \subset w$ and every $n \in \mathbf{Z}$ such that $|n| > m$.*

Proof. For each $\delta, \eta > 0$ and $\bar{w} \subset w$ let $A(\eta, \bar{w}) = \{x \in w: x = \tilde{f}_t(y)$ for some $y \in \bar{w}$ and $|t| \leq \eta\}$. Let $A_\delta(\bar{w}) = \{x \in w: d(x, \bar{w}) \leq \delta\}$. A simple argument using Remark 3.4 shows that given $\delta > 0$ there exists $T > 0$ such that $A_\delta(\bar{w}) \subset A(T, \bar{w})$, for all $\bar{w}$ contained in w. By Proposition 3.1 there exists $\tau \neq 0$ such that $\alpha^n(\bar{w}) = \tilde{f}_{n\tau}(\bar{w})$ for all $n \in \mathbf{Z}$. From this the proof is immediate.

Our next aim is to compare the growth of $d(\bar{w}, x, y)$ with respect to $d(x, y)$. This is done by means of Lemma 3.5 below. First, we will need some more definitions and propositions.

Let $\mu > 0$ and $Q_\mu = \{(x, y) \in M \times M: x, y \in s$ for some $s \in \mathcal{S}$ and $d(s; x, y) = \mu\}$. If we define $\pi: Q_\mu \to M$ by $\pi(x, y) = x$, then using the exponential map we see that, for small μ, π is a continuous locally trivial fibre bundle over M with fibre S^{n-3}. We take such a μ and observe that Q_μ is a compact subset of $M \times M$.

Remark 3.5. It follows directly from hyperbolicity that if $x, y \in M$ are such that $x \neq y$ and $x, y \in s$ for some $s \in \mathcal{S}$, then there exists $t \in \mathbf{R}$ such that $(f_t(x), f_t(y)) \in Q_\mu$. Furthermore, given $T > 0$ there exists $\eta > 0$ such that if $x, y \in M$ are any pair of points which lie in the same strongly stable leaf $s \in \mathcal{S}$ and have the property that $d(s; x, y) > \eta$, then $(f_t(x), f_t(y)) \in Q_\mu$ for some $t > T$.

For each $(x, y) \in Q_\mu$, let $\Omega(x, y)$ denote the set of smooth paths $\alpha:I \to M$ such that $\alpha(0) = x$, $\alpha(1) = y$, and α is homotopic with end-points fixed to a path $\beta:I \to M$ such that $\beta(I) \subset s(x)$.

Given a curve $\alpha:I \to M$ let $\dot\alpha(\tau) = \dot\alpha_s(\tau) + \dot\alpha_u(\tau) + \dot\alpha_1(\tau)$ denote the decomposition of the tangent vector $\dot\alpha(\tau)$ into its components with respect to the splitting $TM = E^s \oplus E^u \oplus E^1$.

For any smooth $\alpha:I \to M$ let

$$\ell(\alpha) = \int_0^1 \| \dot\alpha(\tau) \| \, d\tau,$$

denote its length and let

$$\ell_s(\alpha) = \int_0^1 \| \dot\alpha_s(\tau) \| \, d\tau.$$

PROPOSITION 3.4. *There exists $\delta > 0$ such that any smooth α such that $(\alpha(0), \alpha(1)) \in Q_\mu$ and $\ell_s(\alpha) < \delta$ does not belong to $\Omega(\alpha(0), \alpha(1))$ (i.e., α is not homotopic with end-points fixed to a curve lying entirely in a strongly stable leaf).*

Proof. Let $\bar u \subset \tilde M$ be an unstable leaf in $\tilde M$. Let $\Phi:\bar u \times D^{n-2} \to \tilde M$ be a diffeomorphism onto a tubular neighborhood of $\bar u$ obtained, as usual, via the exponential map. More precisely, let $(V_1(x), \cdots, V_{n-2}(x))$, $x \in \bar u$, be a smooth trivialization of the normal bundle of $\bar u$ by an orthonormal framing. Let $D^{n-2} = \{(t_1, \cdots, t_{n-2}) \in \mathbf{R}^{n-2}: \sum_1^{n-2} t_i^2 \le 1\}$ and let $\Phi:\bar u \times D^{n-2} \to \tilde M$ be defined by

$$\Phi(x, (t_1, \cdots, t_{n-2})) = \exp_x \left(\sum_1^{n-2} \epsilon t_i V_i(x)\right).$$

Then for $\epsilon > 0$ sufficiently small Φ is an embedding onto a closed uniform tubular neighborhood, $\mathcal{V}$, of $\bar u$. It is a simple matter to see that we can choose ϵ small so that for any $x \in \bar u$ the set $A(x) = \{y \in \bar w(x): d(\bar w; x, y) = \mu\}$ is disjoint from $\mathcal{V}$. We go through all this because we want to emphasize the following facts which are easily verified:

There exist constants $k_1, k_2 > 0$ such that for any $x \in$ Interior $(\mathcal{V})$ and $v \in T_x\tilde M$

$$k_1 \| D_x\Phi^{-1}(v) \|_1 \le \| v \| \le k_2 \| D_x\Phi^{-1}(v) \|_1$$

where $\|\cdot\|_1$ denotes the norm with respect to the product metric in $\bar u \times D^{n-2}$. Therefore, there exists $\delta > 0$ such that if $\alpha:I \to \tilde M$ is any curve such that $\alpha(0) \in \bar u$ and $\ell_s(\alpha) < \delta$ where

$$\ell_s(\alpha) = \int_0^1 \| \dot\alpha_s(\tau) \| \, d\tau$$

and $\dot\alpha_s(\tau)$ denotes the strongly stable component of the tangent vector corresponding to the splitting $T\tilde M = \tilde E^s \oplus \tilde E^u \oplus \tilde E^1$; then

$$\alpha(I) \subset \Phi\left(\bar u \times \left(\frac{\epsilon}{2} D^{n-2}\right)\right).$$

Hence, no such path can be the lifting of a path in M which is homotopic, with end points fixed, to a curve lying in a strongly stable leaf.

Since $p(\tilde{u})$ is dense in M, it follows that any curve α such that $\alpha \in \Omega(x, y)$ for some $(x, y) \in Q_\mu$ must necessarily satisfy $\ell_s(\alpha) \geq \delta$.

This proves Proposition 3.4.

PROPOSITION 3.5. *Given $k > 0$ there exists $T > 0$ such that if $(x, y) \in Q_\mu$ and $\alpha \in \Omega(x, y)$ then $\ell(f_{-t}\circ\alpha) > k$ for all $t > T$.*

Proof. Let us choose a riemannian metric for which E^s and $E^u \oplus E^1$ are orthogonal. Naturally, such a metric may only be continuous. Let $\bar{\ell}(\alpha)$ denote the length of the smooth curve α with respect to this new metric. There exist constants $k_1, k_2 > 0$ such that $k_1 \| \bar{v} \| \leq \| v \| \leq k_2 \| \bar{v} \|$, where $\| \quad \|$ denotes our original smooth norm and $\| \bar{\ } \|$ denotes the norm for the new metric. Let $\bar{\ell}_s(\alpha) = \int_0^1 \| \overline{\dot{\alpha}_s(\tau)} \| \, d\tau$. Then there exists $\bar{\delta} > 0$ such that if $(x, y) \in Q_\mu$ and $\alpha \in \Omega(x, y)$ than $\bar{\ell}_s(\alpha) > \bar{\delta}$. Therefore for such α, $\ell(f_{-t}\circ\alpha) \geq k_1\bar{\ell}(f_{-t}\circ\alpha) \geq k_1\bar{\ell}_s(f_{-t}\circ\alpha) \geq K\bar{\delta}e^{\lambda t}$ for suitable positive constants K and λ which are obtained by hyperbolicity and the comparison of norms. Hence, we can take $T > (1/\lambda) \log (k/K\bar{\delta})$.

LEMMA 3.5. *Given $k > 0$, there exists $c > 0$ such that for all $\bar{w} \in \bar{\mathcal{W}}$ and all $x, y \in \bar{w}$ such that $d(\bar{w}; x, y) > c$, we have $d(x, y) > k$.*

Proof. The proof follows directly from Proposition 3.5 and Remark 3.5.

Let $z \in \tilde{M}$ be arbitrary. Since $\tilde{M}$ is simply connected and $w(z)$ is a closed, simply connected smooth submanifold of codimension one in $\tilde{M}$ it follows that there exists two open, connected, simply connected and disjoint sets $V_1(z)$ and $V_2(z)$ such that $\tilde{M} - w(z) = V_1(z) \cup V_2(z)$ and $\partial V_1(z) = \partial V_2(z) = w(z)$. Hence, if z_1 and z_2 lie in the same leaf $\tilde{u} \in \tilde{\mathcal{U}}$ and if $z_1 < z_2$, then there exists an open and connected set $B(z_1, z_2)$ such that $(z_1, z_2) \subset B(z_1, z_2)$ and $\partial B(z_1, z_2) = w(z_1) \cup w(z_2)$. Obviously, $B(z_1, z_2)$ is saturated by the leaves of $\mathcal{W}$. With these notations, together with the definition given below Remark 3.3, we have the following:

PROPOSITION 3.6. *For all $\tilde{u} \in \tilde{\mathcal{U}}$ and all $x, y \in \tilde{u}$ such that $x < y$ we have the identity: $B(x, y) = A(x, y)$.*

Proof. Clearly, $A(x, y) \subset B(x, y)$. Therefore it is only left to prove that if $x_1 \in B(x, y)$ then $w(x_1) \cap (x, y) \neq \emptyset$. Let us assume the contrary. Let $L = \{x_1 \in B(x, y) : w(x_1) \cap (x, y) = \emptyset\}$. Then if $L \neq \emptyset$ we will arrive to a contradiction. If ∂L is the topological boundary of L, then using long product neighborhoods one sees that ∂L is a union of leaves of $\mathcal{W}$ and, by hypothesis $\partial L \neq \emptyset$. Let $z \in \partial L$. It follows immediately from the definitions of $B(x, y)$ and L that $z \in B(x, y)$ and $w(z) \cap [x, y] = \emptyset$. For every $\delta > 0$ there exists $z_1 \neq z$ such that $z_1 \in (z - \delta, z + \delta)$ and $w(z_1) \cap (x, y) \neq \emptyset$. We can assume, without loss of generality, that $z_1 \in (z - \delta, z)$ (the arguments below are exactly the same when $z_1 \in (z, z + \delta)$). Hence, $w(z_2) \cap (x, y) \neq \emptyset$ for all $z_2 \in [z_1, z)$. This follows trivially from the fact that each $w \in \mathcal{W}$ separates $\tilde{M}$ and also from the fact that $A(x, y)$ is connected. Therefore, we have a one-to-one, continuous map $\psi : [z_1, z) \rightarrow (x, y)$ defined by $\psi(z_0) = w(z_0) \cap (x, y)$. That this map is well-defined and one-to-one follows from Remark 3.1. Since $\mathcal{W}$ is a C^1 foliation it follows that, in fact, ψ is a C^1 map. Let $x_1 = \psi(z_1)$ and $x_2 = \sup \psi([z_1, z))$, then $[x_1, x_2] \subset [x, y]$. Let $\varphi: [x_1, x_2] \to \mathbf{R}$ be defined by $\varphi(\bar{z}) = t$ where t is the unique real number such that $\tilde{f}_t(\psi^{-1}(\bar{z})) \in \bar{w}(\bar{z})$. Then using long product neighborhoods one sees that φ is continuous. We claim that under our hypothesis

$$\sup_{\bar{z} \in [x_1, x_2)} |\varphi(\bar{z})| < \infty.$$

If this were not the case, then we could take $\bar{w} \in \bar{\mathcal{W}}$ such that $\bar{w}$ contains a point $\bar{x}$ such that the orbit through $p(\bar{x})$ is periodic or prime period, say, $\tau > 0$ and such that $\bar{w} \cap [x_1, x_2) \neq \emptyset$. We can do this because the periodic orbits are dense in M. If $|\varphi(\bar{z})|$ were unbounded, then since $p(w(\bar{x}))$ is dense in M and $p(w(\bar{x})) = \bigcup_{0 \leq t \leq \tau} f_t(p(\bar{w}(\bar{x})))$; and also because $d([x_1, x_2], [z_1, z]) < \infty$ we would conclude the existence of $N > 0$ such that $d(\bar{w}, \alpha^n(\bar{w})) < N$ for arbitrarily large values of n, where $\alpha \in \pi_1(M)$ is such that $\alpha(w(\bar{x})) = w(\bar{x})$, as given by Proposition 3.1. This would wontradict Proposition 3.3. Hence $|\varphi(\bar{z})|$ is bounded, and it is easy to see that this implies that

$$\sup_{\bar{z} \in [x_1, x_2)} d(\bar{z}, \tilde{f}_{\varphi(\bar{z})}(\psi^{-1}(\bar{z}))) < \infty.$$

On the other hand if $L \neq \emptyset$

$$\sup_{\bar{z} \in [x_1, x_2)} d(\bar{w}(\bar{z}); \bar{z}, \tilde{f}_{\varphi(\bar{z})}(\psi^{-1}(\bar{z})))$$

has to be unbounded, because if it were bounded then using a stable long product neighborhood, relative to a closed disc contained in $\bar{w}(z)$, which contains z, and which has sufficiently large diameter one could prove immediately that $z \in w(x_2)$ which would be a contradiction. Thus if $L \neq \emptyset$ we have a contradiction because Lemma 3.5, implies that

$$\sup_{\bar{z} \in [x_1, x_2)} d(\bar{z}, \tilde{f}_{\varphi(\bar{z})}(\psi^{-1}(\bar{z}))) = \infty.$$

Hence, $A(x, y) = B(x, y)$. Using Proposition 3.6 the following proposition follows easily.

PROPOSITION 3.7. *Let $x_i < y_i$, $i = 1, \cdots, m$ be points in $\tilde{M}$ such that $N = \bigcup_{i=1}^m A(x_i, y_i)$ is connected. Then for any two points $b_1, b_2 \in N$ with $w(b_1) \neq w(b_2)$, there exists an embedding $\mu: I \to \tilde{M}$, transversal to $\mathcal{W}$, such that $\mu(0) = b_1$, $\mu(1) = b_2$ and $\mu(I) \subset N$.*

PROPOSITION 3.8. *Let $w_1, w_2 \in \mathcal{W}$ be such that $w_1 \neq w_2$. Then, there exists an embedding $\mu: I \to \tilde{M}$ such that μ is transversal to $\mathcal{W}$, $\mu(0) \in w_1$, and $\mu(1) \in w_2$. Furthermore any two such embeddings meet exactly the same leaves of $\mathcal{W}$.*

Proof. Fix $x, y \in \tilde{M}$ with $x < y$. Let $x_1 \in w_1$ and $x_2 \in w_2$. Let $\delta: I \to \tilde{M}$ be a path such that $\delta(0) = x_1$ and $\delta(1) = x_2$. Then, by Corollary 3.2, compactness, and connectedness of I, there exists $\alpha_1, \cdots, \alpha_m \in \pi_1(M)$ such that $N = \bigcup_{i=1}^m \alpha_i(A(x, y))$ is connected and $\delta(I) \subset N$. By Proposition 3.7, there exists $\mu: I \to \tilde{M}$, transversal to $\mathcal{W}$, such that $\mu(0) = x_1$ and $\mu(1) = x_2$. That any two smooth paths, transversal to $\mathcal{W}$, which join w_1 with w_2 must meet the same leaves of $\mathcal{W}$ is an easy consequence of Proposition 3.6.

We can introduce a total order in $\mathcal{W}$. We say that $w_1 < w_2$ if there exists a positively oriented transversal $\mu: I \to \tilde{M}$ such that $\mu(0) \in w_1$ and $\mu(1) \in w_2$. The relation "$\leq$" is obviously reflexive, transitive and antisymmetric by the previous propositions.

PROPOSITION 3.9. *There exists a smooth embedding* $h: \mathbf{R} \to \tilde{M}$ *such that* $h(\mathbf{R})$ *intersects, transversally, every leaf of* $\mathcal{W}$.

Proof. Let $\mathcal{E}$ be the set of all embeddings $h: \mathbf{R} \to \tilde{M}$ which are transversal to $\mathcal{W}$. We introduce a preordering in $\mathcal{E}$ as follows: we say $h_1 \leq h_2$, if for every $t_1 \in \mathbf{R}$ there exists $t_2 \in \mathbf{R}$ such that $w(h_1(t_1)) = w(h_2(t_2))$. Given a totally ordered subset of $\mathcal{E}$ it is easy to construct an embedding $h: \mathbf{R} \to \tilde{M}$ which is an upper bound for that subset. Thus, by virtue of Zorn's Lemma, there exists a maximal element $h \in \mathcal{E}$. This maximal embedding meets all leaves of $\mathcal{W}$.

COROLLARY 3.3. There exists a C^1 submersion $g: \tilde{M} \to \mathbf{R}$ such that $\mathcal{W} = \{g^{-1}(t) : t \in \mathbf{R}\}$. That is to say, $\mathcal{W}$ comes from a C^1 submersion onto the real line.

Proof. Let $\mathcal{L} = h(\mathbf{R})$ where $h: \mathbf{R} \to \tilde{M}$ is an embedding that meets, transversally, every leaf of $\mathcal{W}$. Let $g: \tilde{M} \to \mathbf{R}$ be given by $g(x) = h^{-1}(w(x) \cap \mathcal{L})$. Then g satisfies the requirements in Corollary 3.3. because $\mathcal{W}$ is a C^1 foliation.

COROLLARY 3.4. *There exists a monomorphism* $k: \pi_1(M) \to \mathit{Diff}^1(\mathbf{R})$ *given by* $K(\alpha) = g \circ \alpha \circ h$ *where h and g are as in the previous corollary. For each* $\alpha \in \pi_1(M)$, $k(\alpha): \mathbf{R} \to \mathbf{R}$ *is a* C^1 *diffeomorphism that has either no fixed points or else the fixed points are generic.*

Proof. That k is a homomorphism follows from the fact that $\pi_1(M)$ acts on $\mathcal{W}$. If the kernel of k were nontrivial then every leaf of $\mathcal{W}$ would not be simply connected, which is absurd. If $k(\alpha)$ has a fixed point then due to the holonomy of $\mathcal{W}$ this fixed point has to be generic.

By Corollary 3.3, there exists a C^1 submersion $g: \tilde{M} \to \mathbf{R}$ which induces $\mathcal{W}$. Next we show that each compact set in $\tilde{M}$ is contained in an open subset of $\tilde{M}$ which is diffeomorphic to euclidean space. We will also show that g is locally trivial. All of this is accomplished by means of the global hyperbolicity of $\tilde{f}_t$.

We recall that we had provided M with a fixed, smooth, riemannian metric $\langle\ ,\ \rangle$. Let ∇ be the smooth riemannian connection given by this metric. The connection ∇ induces a connection $\nabla(\hat{s})$, in each leaf $\hat{s} \in \hat{\mathbb{S}}$. We have a C^1 map $H_\delta: G_\delta \to M$, where $G_\delta = \{v \in E^s \oplus E^1 : \| v \| \leq \delta\}$ is the δ-disc bundle associated with the C^1 riemannian bundle $E^s \oplus E^1$ and where the riemannian metric in $E^s \oplus E^1$ is the one induced by $\langle\ ,\ \rangle$. This map is given explicitly by

$$H_\delta(v(x)) = \exp_x(v(x)),$$

where $v(x)$ is a vector in the fibre over x, and $\exp_x$ is the exponential map at x. of the leaf $\hat{s}(x)$, with respect to $\nabla(\hat{s}(x))$. For $\delta > 0$ sufficiently small H_δ maps, for each $x \in M$, the ball

$$B_\delta(x) = \{v(x) \in E_x^s \oplus E_x^1 : \| v(x) \| \leq \delta\}.$$

diffeomorphically onto a closed ball which lies in $\hat{s}(x)$ and contains x in its interior. Let $\overline{\langle\ ,\ \rangle}$ be the metric in $\tilde{M}$ which is the lifting of $\langle\ ,\ \rangle$, to the universal covering $p:\tilde{M} \to M$. Then p is a local isometry with respect to these two metrics. Let $\tilde{E}^s \oplus \tilde{E}^1$ be the bundle in $\tilde{M}$ which covers $E^s \oplus E^1$ and, so, is tangent to $\mathcal{W}$. Using a similar definition as that of H_δ, we obtain a C^1 map $F_\delta:\tilde{G}_\delta \to \tilde{M}$, where $\tilde{G}_\delta = \{v \in \tilde{E}^s \oplus \tilde{E}^1 : \| \bar{v} \| \leq \delta\}$. Here $\| ^- \|$ denotes the norm with respect to $\overline{\langle\ ,\ \rangle}$, and we are taking δ as above. Clearly, $p \circ F_\delta = H_\delta \circ (Dp \mid \tilde{G}_\delta)$. Then, F_δ maps

$$B_\delta(\tilde{x}) = \{v(\tilde{x}) : v(\tilde{x}) \in \tilde{E}_{\tilde{x}}^s \oplus \tilde{E}_{\tilde{x}}^1 : \| \overline{v(\tilde{x})} \| \leq \delta\}.$$

diffeomorphically into a closed ball in $w(\tilde{x})$.

Let $h:\mathbf{R} \to \tilde{M}$ be an embedding which meets, transversally, every leaf of $\mathcal{W}$. Let $T:\mathbf{R} \times \mathbf{R} \to \tilde{M}$ be defined by $T(t_1, t_2) = \tilde{f}_{t_2}(h(t_1))$. Then T is a proper embedding of $\mathbf{R}^2$ into $\tilde{M}$ which is also transversal to $\mathcal{W}$. Let

$$P_\delta = \{v(\tilde{x}) \in \tilde{G}_\delta : \tilde{x} \in \tilde{M}, \overline{\langle v(\tilde{x}), \tilde{X}(\tilde{x}) \rangle} = 0\}.$$

where $\tilde{X}(\tilde{x}) = (d/dt)(\tilde{f}_t(\tilde{x}))\,|_{t=0}$. Then, P_δ is a C^∞ bundle over $\tilde{M}$ with typical fibre the closed $(n-2)$-disc,

$$D^{n-2} = \{a \in \mathbf{R}^{n-2} : \| a \| \leq 1\}.$$

and projection $\pi(v(\tilde{x})) = \tilde{x}$. Let $K_\delta = \{v(\tilde{x}) \in P_\delta : \tilde{x} \in T(\mathbf{R}^2)\}$. Then K_δ is the total space of the bundle $\pi \mid K_\delta : K_\delta \to T(\mathbf{R}^2)$. Since $T(\mathbf{R}^2)$ is contractible this bundle is trivial. Hence, K_δ is diffeomorphic to $\mathbf{R}^2 \times D^{n-2}$. Let $F = F_\delta \mid K_\delta$. For $\delta > 0$ sufficiently small, F is a C^1 embedding. Therefore, $F(K_\delta)$ is also C^1 diffeomorphic to $\mathbf{R}^2 \times D^{n-2}$. We denote by A_δ the interior of $F_\delta(K_\delta)$. We have that A_δ is C^1 diffeomorphic to elucidean space through a C^1 diffeomorphism that takes the foliation induced by $\mathcal{W}$ in A_δ to the foliation in $\mathbf{R}^n$ whose leaves are the hyperplanes $x_1 =$ constant.

If we let $\gamma_{t_1} = \{T(t_1, t_2) : t_2 \in \mathbf{R}\}$, then $N_{t_1} = F(K_\delta) \cap w(h(t_1))$ is a closed tubular neighborhood of the curve γ_{t_1}, considered as a submanifold of $w(h(t_1))$. In fact, what we are just doing is to obtain a tubular neighborhood, in $w(h(t_1))$, by means of Fermi coordinates.

It follows from the compactness of M and from the fact that $p:\tilde{M} \to M$ is a local isometry, that N_{t_1} is a uniform tubular neighborhood, in $w(h(t_1))$, of the curve γ_{t_1}. Hence, for any point $\tilde{x} \in w(h(t_1))$ there exists $t_0 > 0$ such that $\tilde{f}_t(\tilde{x}) \in N_{t_1}$, for all $t > t_0$. To see this consider the strongly stable leaf through $\tilde{x}$, $\tilde{w}(\tilde{x})$, and let $\tilde{y} = \tilde{w}(\tilde{x}) \cap \gamma_{t_1}$. Then since $\lim_{t\to\infty} d(\tilde{f}_t(\tilde{x}), \tilde{f}_t(\tilde{y})) = 0$, it follows that for some $t_0 > 0$, $\tilde{f}_t(\tilde{x})$ belongs to the ϵ-neighborhood of γ_{t_1}, for all $t > t_0$.

PROPOSITION 3.10. *Every compact set $K \subset \tilde{M}$ is contained in an open subset which is diffeomorphic to euclidean space.*

Proof. Let $K \subset \tilde{M}$ be any compact set. By the above remarks, for every $\tilde{x} \in \tilde{M}$ there exists $t(x) > 0$ such that $\tilde{f}_t(\tilde{x})$ is contained in A_δ for all $t > t(\tilde{x})$. Since K is compact, there exists $T > 0$ such that $\tilde{f}_T(K) \subset A_\delta$. Therefore $K \subset \tilde{f}_{-T}(A_\delta)$. The proposition follows since $\tilde{f}_{-T}(A_\delta)$ is diffeomorphic to euclidean space.

We recall the following lemma of Brown and Stallings [6] [31]: Let M be a paracompact manifold such that every compact subset is contained in an open set diffeomorphic to euclidean space. Then M itself is diffeomorphic to euclidean space.

THEOREM 3.1. *If $f_t: M \to M$ is a codimension one Anosov flow on the compact manifold M, then the universal covering of M is diffeomorphic to euclidean space.*

The following is a sharper result:

THEOREM 3.2. *There exists a C^1-diffeomorphism $f: \tilde{M} \to \mathbf{R}^n$, taking leaves of $\mathcal{W}$ onto the hyperplanes $x_1 =$ constant.*

Proof. To prove the theorem it suffices to show that there exists a complere nonsingular, smooth vector field defined in all of $\tilde{M}$, which is transversal to $\mathcal{W}$, and such that every orbit of this vector field meets every leaf of $\mathcal{W}$. This is accomplished by constructing a suitable vector field in A_δ and then "blowing up" this vector field to all of $\tilde{M}$, by means of $\tilde{f}_t$.

Since M is compact and f_t Anosov, it is easy to see that there exists $\tau > 0$ such that $\tilde{f}_\tau(\bar{A}_\delta) \subset A_{\delta/2}$, where $\bar{A}_\delta$ denotes the closure of A_δ. Then, $N = \bar{A}_\delta - f_\tau(\bar{A}_\delta)$, is a fundamental domain, in the sense that for every $x \in \tilde{M} - T(\mathbf{R}^2)$ there exists a unique integer m such that $\tilde{f}_{m\tau}(x) \in N$. Then, via $F: K_\delta \to \tilde{M}$, and using the fact that F is essentially a C^1 diffeomorphism from $\mathbf{R}^2 \times D^{n-2}$ into $\tilde{M}$ which takes $\{\text{point}\} \times D^{n-2}$ into leaves of $\mathcal{W}$, one can easily construct a complete, smooth vector field Y in a neighborhood of the smooth manifold with boundary $\bar{A}_\delta$ with the following properties:

i) Y is transversal to $\mathcal{W}$

ii) Every orbit of Y meets every leaf of $\mathcal{W}$

iii) There exists a neighborhood, $\mathcal{V}$, of $\partial(\tilde{f}_\tau(\bar{A}_\delta))$ such that for every $x \in \mathcal{V}$

$$Y(x) = D\tilde{f}_\tau(Y(f_{-\tau}(x))).$$

Define the vector field $Z: \tilde{M} \to T\tilde{M}$ by

$$Z(x) = \begin{cases} Y(x) \quad \text{if} \quad x \in \bar{A}_\delta \\ D\tilde{f}_{m\tau}(Y(\tilde{f}_{-m\tau}(x))) \text{ if } x \notin \bar{A}_\delta \text{ and } m \text{ is the unique integer such that} \\ \tilde{f}_{m\tau}(x) \in N. \end{cases}$$

It follows directly from the constructions above that Z is a well-defined, complete, nonsingular smooth vector field such that every orbit of Z meets, transversally, every leaf of $\mathcal{W}$. Thus, we have proved the theorem.

We observe that Theorem 3.2 implies immediately that $g: \tilde{M} \to \mathbf{R}$ is a locally trivial fibre bundle.

Remark 3.6. Theorem 3.1 is false if the Anosov flow is not codimension one. For example, if $T_1(M)$ is the unit sphere bundle of a compact, smooth manifold M with negative sectional curvature and if $n = \dim M > 2$, then, since

$$\pi_{n-1}(T_1(M)) = \mathbf{Z},$$

$T_1(M)$ cannot be covered by euclidean space.

We have the following obvious corollary:

COROLLARY 3.5. *If* $n = \dim M \leq 4$ *and* M *admits an Anosov flow then* $\tilde{M} = \mathbf{R}^n$.

COROLLARY 3.6. *Let* $f_t: M \to M$ *be a codimension one Anosov flow on the compact, connected, smooth manifold* M. *Then*

1) M *is an Eilenberg-MacLane space* $K(\pi_1(M), 1)$
2) $\pi_1(M)$ *has finite cohomological dimension.*
3) $\pi_1(M)$ *has no elements of finite order.*
4) *Let* $\dim M = n$. *Then, every locally-flat embedding* $f: S^{n-1} \to M$ *can be extended to an embedding of the closed n-disc,* $H: D^n \to M$. *Therefore* M *is irreducible.*
5) *If* $x \in M$ *is such that* $\hat{s}(x)$ *does not contain a periodic orbit then any* $\alpha \in \pi_1(M, x)$ *can be represented by a smooth curve transversal to* $\hat{\mathcal{S}}$.

Proof. We will only prove 4), since 1), 2) and 3) are standard (see [17]) and 5) follows from the fact that $\alpha(w) \neq w$ for all $\alpha \in \pi_1(M)$ where $w \in \mathcal{W}$ denotes a lifting, to $\tilde{M}$, of $\hat{s}(x)$.

By Theorem 3.1 there exists a covering map $\pi: \mathbf{R}^n \to M$. We recall that a topological embedding $f: S^{n-1} \to M^n$ is said to be *locally-flat,* if for every $x \in S^{n-1}$ there exists a neighborhood $\mathcal{V}$ of $f(x)$ in M^n, and a homeomorphism of pairs

$$h: (\mathcal{V}, \mathcal{V} \cap f(S^{n-1})) \to (\mathbf{R}^n, \mathbf{R}^{n-1}). \qquad \text{(See [18])}$$

Now, let $f: S^{n-1} \to M$ be a locally flat embedding. Since $n \geq 3$ by hypothesis, there exists an embedding $\bar{f}: S^{n-1} \to \mathbf{R}^n$ such that $\pi \circ \bar{f} = f$. We have that $\bar{f}$ is also locally flat since π is a local homeomorphism. By Schoenflies Theorem (M. Brown [5]), $\bar{f}$ extends to a topological embedding $F: D^n \to \mathbf{R}^n$. Then $F(D^n)$ projects into M, under π, in a one-to-one fashion. Otherwise, there would exist a nontrivial $\alpha \in \pi_1(M)$ such that $\alpha(F(D^n)) \cap F(D^n) \neq \emptyset$. Therefore, either $\alpha(F(D^n)) \subset F(D^n)$ or else $\alpha^{-1}(F(D^n)) \subset F(D^n)$. By Brower's Fixed Point Theorem, α would have a fixed point. This would be absurd. Thus 4) is proven by setting $H = \pi \circ F$.

Now we will assume that $f_t: M \to M$ is a codimension one Anosov flow, such that $\hat{\mathcal{S}}$ is a C^2 foliation. Then, $\mathcal{W}$ is also a C^2 foliation. By Corollary 3.4, there exists an injective representation $k: \pi_1(M) \to \mathrm{Diff}^2(\mathbf{R})$. In the following theorem we will think of each $\alpha \in \pi_1(M)$ as a C^2-diffeomorphism $\alpha: \mathbf{R} \to \mathbf{R}$. In all that follows L will denote the center of $\pi_1(M)$.

THEOREM 3.3. *The center of* $\pi_1(M)$ *is either trivial or else it is free cyclic.*

Proof. Suppose L is non-trivial and let $\alpha \in L$ be a non-trivial element. Then α cannot have a fixed point. Let us suppose the contrary and let $\alpha(t) = t$. Then considering either α or α^{-1}, we can assume that $\dot{\alpha}(t) < 1$ and that $U \subset \mathbf{R}$ is a neighborhood of t such that $\bigcap_{n \geq 0} \alpha^n(U) = \{t\}$. Since the periodic orbits are dense in M, there exists a nontrivial $\beta \in \pi_1(M)$, and $\bar{t} \in U$, $\bar{t} \neq t$, such that $\beta(\bar{t}) = \bar{t}$. This contradicts Kopell's Lemma 1 (a) in [19]. Therefore α has no fixed points. Let $\gamma \in \pi_1(M)$ and $t \in \mathbf{R}$ be such that $\gamma(t) = t$ and let $L(t)$ be the orbit of t

by the center of $\pi_1(M)$. Then for any $\bar{t} \in L(t)$, $\gamma(\bar{t}) = \bar{t}$. It is easy to see that if $L \neq \mathbf{Z}$, then $L(t)$ is dense in $\mathbf{R}$. Since γ has only isolated fixed points, it follows that L must be isomorphic with $\mathbf{Z}$.

COROLLARY 3.7. *If $L \simeq \mathbf{Z}$, then there exists a monomorphism $\bar{k}:\pi_1(M)/L \to$ Diff^2 (S^1) such that if $\bar{k}(\bar{\alpha}): S^1 \to S^1$ has a periodic point, then either $\bar{k}(\bar{\alpha})$ is structurally stable or else $\bar{\alpha}$ is an element of finite order in $\pi_1(M)/L$. Furthermore, $\pi_1(M)/L$ is not abelian.*

Proof. Let $\alpha: \mathbf{R} \to \mathbf{R}$ be a generator of L. Since α has no fixed points we may assume, using a new reparametrization of $\mathbf{R}$ if necessary, that α is the translation $\alpha(t) = t + 1$. For each $\bar{\beta} \in \pi_1(M)/L$, we define $\bar{k}(\bar{\beta}): S^1 \to S^1$ by $\bar{k}(\bar{\beta})(e^{2\pi i t}) = e^{2\pi i \beta(t)}$, where $\beta: \mathbf{R} \to \mathbf{R}$ is a representative in the coset $\bar{\beta}$. It is easy to verify that $\bar{k}$ is well defined and a monomorphism. The rest of the corollary follows from the holonomy properties of $\hat{\mathfrak{S}}$ and Kopell's Lemma.

THEOREM 3.4. *If* dim $M = 3$ *and* $L = \mathbf{Z}$ *then M admits an effective action of S^1 without fixed points. If $\pi_1(M)/L$ is torsion free, then $\pi_1(M)/L$ is isomorphic to the fundamental group of a compact surface M^2, of genus greater than one. Furthermore if $a \in H^2$ $(M^2, \mathbf{Z})$ corresponds to the central extension $a: 0 \to \mathbf{Z} \to \pi_1(M) \to \pi_1(M^2) \to 0$, then M is diffeomorphic to the principal circle bundle, ξ, associated to a. Hence, M admits a principal circle action. Both foliations $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{U}}$ can be made transversal to the orbits of this circle action by differentiable isotopies of M and there is an inequality $|\chi(\xi)| \leq |\chi(M^2)|$, where $\chi(\xi)$ and $\chi(M^2)$ denote the Euler Characteristics of the circle bundle and M^2, respectively.*

Proof. By Waldhausen ([35], [28]) every irreducible, orientable, closed three manifold, which is aspherical and with non-trivial center in its fundamental group, admits a smooth and effective action of S^1. By the results of Conner and Raymond [7] [28] it follows that if $L \approx \mathbf{Z}$, then this action must be principal and that $M \to M/S^1$ is classified by the extension $a \in H^2$ $(M^2, \mathbf{Z})$. William Thurston's Thesis [33] says that under our hypothesis the foliations $\hat{\mathfrak{S}}$ and $\hat{\mathfrak{U}}$ can be made transversal to the orbits of this principal circle action. The inequality follows from a result of Wood [38].

Remarks 3.7. Theorem 3.1 says that if $f_t: M \to M$ is a codimension one Anosov flow, with M compact, then $\pi_1(M)$ is a uniform space form. It is not known which discrete groups can act freely, properly discontinuously and uniformly in $\mathbf{R}^n$ (see Wall [36] for space form problems). It is known [36] that a free poly-cyclic group of rank n (a P-group of rank n, [36]) acts freely, properly discontinuously and uniformly in $\mathbf{R}^n$.

We conjecture that if $\pi_1(M^{n+1})$ is a P-group and if M^{n+1} admits a codimension one Anosov flow $f_t: M^{n+1} \to M^{n+1}$ then f_t is topologically conjugate to the suspension of a codimension one hyperbolic toral isomorphism. Clearly, if $f_t: M^{n+1} \to M^{n+1}$ is topologically conjugate to a hyperbolic toral isomorphism $f: T^n \to T^n$, then, since $\pi_1(M) \approx \mathbf{Z}^n \times_f \mathbf{Z}$ (semidirect product), $\pi_1(M)$ is a P-group of rank n.

If dim $M = 3$ and if $\pi_1(M)$ is a P-group then it follows from a theorem of Stallings [30] (we recall that M is irreducible) that M fibres over S^1 with the torus T^2 as fibre. When dim $M > 5$ and $\pi_1(M)$ is a P-group it follows from Farrell's thesis ([8] [36]) that M fibres over S^1. Thus the conjecture seems to be true. We also conjecture that Theorem 3.4 remains true when dim $M > 3$. That is to say, if $\pi_1(M)/L \simeq \mathbf{Z}$ then M admits an effective, smooth action of S^1. Theorem II.7 of [11] somewhat supports this conjecture.

§4. Existence of global cross-sections for codimension one Anosov flows

In this chapter we give a necessary and sufficient condition for the existence of a smooth global cross-section for a codimenison one Anosov flow $f_t: M \to M$. The conditions are given in terms of the first integral homology group of M, and also in terms of the way in which the periodic orbits, oriented by the flow, and considered as integral 1-cycles, enter into this group.

Since the periodic orbits are dense in M, one expects, intuitively, that if the periodic orbits are, homologically positive multiples of a particular one, then the flow admits a global cross-section. This is the germ of the idea that led us to Theorem 4.1.

If a codimension one Anosov flow $f_t: M \to M$ admits a global cross-section Σ^{n-1}, then the Poincaré map $f: \Sigma^{n-1} \to \Sigma^{n-1}$ induced on this cross-section is a codimension one Anosov diffeomorphism. Since $\Omega(f_t) = M$ we have that $\Omega(f) = \Sigma^{n-1}$. Then it follows from Franks (Theorem 6.3 of [9]), that Σ^{n-1} is homeomorphic to the $(n - 1)$-torus, $T^{n-1} = S^1 \times \cdots \times S^1$, and that f is topologically conjugate to the hyperbolic toral isomorphism induced by $f_*: H_1(\Sigma^{n-1}, \mathbf{Z}) \to H_1(\Sigma^{n-1}, \mathbf{Z})$. From this we conclude that if a codimension one Anosov flow admits a global cross-section Σ^{n-1}, then it is topologically equivalent to the suspension of a hyperbolic toral isomorphism.

Definition 4.1. Given a diffeomorphism $f: N^n \to N^n$, let $N^{n+1}(f)$ be the smooth manifold obtained as the quotient space of $N^n \times \mathbf{R}$ under the free and properly discontinuous action of $\mathbf{Z}$ given by $\varphi_m(x, t) = (f^m(x), t + m)$, $m \in \mathbf{Z}$. Let $\Psi_t(f): N^{n+1}(f) \to N^{n+1}(f)$ be the flow induced by the flow $\varphi_t(x, s) = (x, t + s)$. Then, $(\Psi_t(f); N^{n+1}(f))$ is called the *suspension flow* associated with f.

Definition 4.2. Let $g_t: N^n \to N^n$ be a nonsingular, smooth flow on the compact, connected smooth n-manifold N. A compact, connected, codimension one smooth submanifold $\Sigma^{n-1} \subset N^n$ is called a *cross-section* for g_t if

1) Σ^{n-1} meets transversally the flow.
2) For every $x \in \Sigma^{n-1}$, there exists $t(x) > 0$ such that $g_{t(x)}(x) \in \Sigma^{n-1}$.

If Σ^{n-1} is a cross-section for $g_t: N^n \to N^n$, then there exists a smooth reparametrization $\hat{g}_t: N^n \to N^n$ such that $\hat{g}_1(\Sigma^{n-1}) = \Sigma^{n-1}$. Then $\hat{g}_1: \Sigma^{n-1} \to \Sigma^{n-1}$ is a diffeomorphism called the Poincaré map of $(g_t; \Sigma^{n-1}; N^n)$. For each $x \in N^n$ there exists a unique $t(x) \in (0, 1]$ such that $\hat{g}_{t(x)}(x) \in \Sigma^{n-1}$, and the smooth map $F: N^n \to S^1$ given by $F(x) = e^{2\pi i t(x)}$ is a locally trivial submersion. Therefore, if $g_t: N^n \to N^n$ admits a cross-section Σ^{n-1}, then N^n fibres over S^1 with fibre

Σ^{n-1}, and N^n is obtained from $\Sigma^{n-1} \times I$ by attaching differentiably the ends through ϑ_1. As a differentiable manifold, N^n depends only on Σ^{n-1} and the pseudo-isotopy class of ϑ_1. Furthermore, $\pi_1(N^n) = \pi_1(\Sigma^{n-1}) \times_\varphi \mathbf{Z}$ (semidirect product), where $\varphi: \mathbf{Z} \to Aut\ (\pi_1(\Sigma^{n-1}))$ is given by

$$\varphi(m) = (\vartheta_1^{\,m})_*: \pi_1(\Sigma^{n-1}) \to \pi_1(\Sigma^{n-1}).$$

If $f: N^n \to N^n$ is a diffeomorphism and $(\Psi_t(f), N^{n+1}(f))$ its suspension flow and if $\pi: N^n \times \mathbf{R} \to N^{n+1}(f)$ denotes the quotient map, then $\pi(N^n \times \{0\})$ projects onto a cross-section for $\Psi_t(f)$. We have a natural fibering $F_f: N^{n+1}(f) \to S^1$ given by $F_f(\pi(x, t)) = e^{2\pi i t}$.

If a flow $g_t: N^n \to N^n$ admits a cross-section Σ^{n-1}, with Poincaré map $\vartheta_1: \Sigma^{n-1} \to \Sigma^{n-1}$, then there exists a diffeomorphism $h: N^n \to \Sigma^n(g_1)$ such that the following diagram commutes:

$$\begin{array}{ccc} N^n & \xrightarrow{\ h\ } & \Sigma^n(\vartheta_1) \\ \Big\downarrow g_t & & \Big\downarrow \psi_t(\vartheta_1) \\ N^n & \xrightarrow{\ h\ } & \Sigma^n(\vartheta_1) \end{array}$$

THEOREM 4.1. *Let $f_t: M^n \to M^n$ be a codimension one Anosov flow on the compact, connected, orientable smooth manifold M. Then f_t is topologically equivalent to the suspension of a hyperbolic toral isomorphism $A: T^{n-1} \to T^{n-1}$ if and only if rank $(H_1(M, \mathbf{Z})) = 1$, and the periodic orbits represent non-trivial elements in the free part of $H_1(M, \mathbf{Z})$.*

Proof of necessity. If $f_t: M^n \to M^n$ is topologically equivalent to the suspension of $A: T^{n-1} \to T^{n-1}$, then $\pi_1(M) = \mathbf{Z}^{n-1} \times_A \mathbf{Z}$. That is to say, $\pi_1(M)$ is the semidirect product of $\mathbf{Z}^{n-1}$ with $\mathbf{Z}$, via the homomorphism $\varphi: \mathbf{Z} \to Aut(\mathbf{Z}^{n-1})$ given by $\varphi(m) = A^m: \mathbf{Z}^{n-1} \to \mathbf{Z}^{n-1}$. In other words, $\pi_1(M)$ consists of pairs $(\bar{a}; m)$ where $\bar{a} = (a_1, \cdots, a_{n-1}) \in \mathbf{Z}^{n-1}$, $m \in \mathbf{Z}$; and multiplication is given by $(\bar{a}_1; m_1)\ (\bar{a}_2; m_2) = (\bar{a}_1 + A^{m_1}(\bar{a}_2); m_1 + m_2)$. The subgroup $G = \{(\bar{a}; 0): \bar{a} \in \mathbf{Z}^{n-1}\}$ is normal. Since $\pi_1(M)/G \approx \mathbf{Z}$ it follows that $[\pi_1(M), \pi_1(M)] \subset G$. On the other hand, let $(\bar{a}; 0) \in G$ be arbitrary. We want to solve in the group $\pi_1(M)$ the equation

$$(\bar{a}; 0)^m = (\bar{0}; 1)\ (\bar{b}; 0)\ (\bar{0}; -1)\ (-\bar{b}; 0).$$

where $\bar{0} = (0, \cdots, 0)$ and $m \in \mathbf{Z}$. That is to say, we want to find $m \in \mathbf{Z}$ and $\bar{b} \in \mathbf{Z}^{n-1}$, such that $(\bar{a}; 0)^m$ is the simple commutator $[(\bar{0}; 1), (\bar{b}; 0)]$. This is equivalent to finding $m \in \mathbf{Z}$ and $b \in \mathbf{Z}^{n-1}$ such that $m\bar{a} = (A - I)\bar{b}$. Since A is hyperbolic, it follows that $A - I$ is invertible in the rationals and from this one obtains the required m and $\bar{b}$. From this it follows immediately that rank $(H_1(M, \mathbf{Z})) = 1$. Since f_t is topologically equivalent to a suspension, there exists $F: M \to S^1$ such that restricted to any periodic orbit, has positive degree. Hence, all periodic orbits represent positive multiples of a suitable generator of the free part of $H_1(M, \mathbf{Z})$.

Proof of sufficiency. Conversely, suppose that rank $(H_1(M, \mathbf{Z})) = 1$ and that all periodic orbits represent non trivial elements of the free part of $H_1(M, \mathbf{Z})$. Let $\Psi:\pi_1(M, x_0) \to \mathbf{Z}$ be the homomorphism given by $\Psi = \pi\circ\varphi$ where $\varphi:\pi_1(M, x_0) \to H_1(M, \mathbf{Z})$ is the Hurewicz homomorphism and $\pi:H_1(M, \mathbf{Z}) \to \mathbf{Z}$ is the projection homomorphism onto the free part of $H_1(M, \mathbf{Z})$. Then, since S^1 is aspherical, there exists a differentiable map $g:(M, x_0) \to (S^1, 1)$ such that $g_*:\pi_1(M, x_0) \to \pi_1(S^1, 1)$, is equal to Ψ and such that g restricted to any periodic orbit has nonzero degree. Furthermore, as g_* is an epimorphism, we can assume that $1 \in S^1$ is a regular value of g and that $N^{n-1} = g^{-1}(1)$ is a connected submanifold of M. That we can choose g such that $g^{-1}(1)$ is connected is contained in the proof of the fibration theorem of Browder and Levine [4], and it also follows by the methods of Stallings [30].

Since N^{n-1} does not disconnect M, we can "split" M along N^{n-1}. That is to say, there exists a smooth, connected manifold W whose boundary, ∂W, consists of two submanifolds N_1 and N_2 each one diffeomorphic to N^{n-1}; and if we "glue", differentiably, N_1 and N_2 by means of a diffeomorphism $h:N_1 \to N_2$ then the resulting smooth manifold $\bar{M}$ is diffeomorphic to M by a diffeomorphism $F:\bar{M} \to M$ such that $F(N_1) = F(N_2) = N^{n-1}$. Let $\tilde{M}$ be the smooth manifold, without boundary, obtained from $W \times \mathbf{Z}$, by identifying $N_2 \times \{m\}$ with $N_1 \times \{m+1\}$ via the diffeomorphism $h_m(x, m) = (h(x), m + 1)$, for each $m \in \mathbf{Z}$. If $q:W \times \mathbf{Z} \to \tilde{M}$ denotes the quotient map of this identification, then $\tilde{M}$ is the union of the manifolds $W_i = q(W \times \{i\})$, $i \in \mathbf{Z}$. There exists a natural diffeomorphism $\alpha:\tilde{M} \to \tilde{M}$ such that $\alpha(W_i) = W_{i+1}$ and such that the free and properly discontinuous action generated by this diffeomorphism has orbit space diffeomorphic with $\bar{M}$. Furthermore, if $\bar{p}:\tilde{M} \to \bar{M}$ denotes the projection then $p:\tilde{M} \to M$ defined by $p = F\circ\bar{p}$ is the *regular infinite cyclic covering* associated with $\mathrm{Ker}(\Psi)$ and, therefore, its group of deck transformations is isomorphic to $\mathbf{Z}$ and we may assume that the action is given by α. Furthermore, there exists a smooth map $\tilde{g}:\tilde{M} \to \mathbf{R}$ such that the diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{g}} & \mathbf{R} \\ \downarrow p & & \downarrow \exp \\ M & \xrightarrow{g} & S^1 \end{array}$$

is commutative.

If we provide $\tilde{M}$ with the riemannian metric which is the lifting of one in M, and if $\tilde{f}_t:\tilde{M} \to \tilde{M}$ denotes the lifting of f_t, then $\tilde{f}_t$ is a codimension one Anosov flow on $\tilde{M}$.

Since $\exp_*\circ\tilde{g}_* = g_*\circ p_*:H_1(\tilde{M}, \mathbf{Z}) \to \mathbf{Z}$ is the trivial homomorphism, we see that $\tilde{f}_t$ does not contain a periodic orbit (if $\tilde{f}_t$ had a periodic orbit, then this periodic orbit would cover a periodic orbit in M which would represent an element of torsion in $H_1(M, \mathbf{Z})$). Hence, both the alpha and omega limit-sets of $\tilde{f}_t$ are empty. Otherwise, the proof of M. Hirsch in Proposition 1.7 of [9], for diffeomorphisms, can be adapted to show the existence of a periodic orbit for $\tilde{f}_t$. Therefore, for every $\tilde{x} \in \tilde{M}$ and every integer m, there exists $t(m, \tilde{x}) > 0$ such that $\tilde{f}_t(\tilde{x}) \notin \bigcup_{i=-m}^{m} W_i$, for all t such that $|t| > t(m, \tilde{x})$. This means, precisely, that for every $\tilde{x} \in \tilde{M}$, $\lim_{t\to\infty} \tilde{g}(\tilde{f}_t(\tilde{x}))$ is equal to either ∞ or $-\infty$.

Let $A_+ = \{\tilde{x} \in \tilde{M}: \lim_{t\to\infty} \tilde{g}(\tilde{f}_t(\tilde{x})) = \infty\}$ and $A_- = \{\tilde{x} \in \tilde{M}: \lim_{t\to\infty} \tilde{g}(\tilde{f}_t(\tilde{x})) = -\infty\}$, then $A_+ \cup A_- = \tilde{M}$ and $A_+ \cap A_- = \emptyset$. But both A_+ and A_- are of the second category of Baire which can be easily obtained from the fact that both A_+ and A_- are saturated by the stable leaves of the flow $\tilde{f}_t$, and whose projections are dense in M. Hence, either $A_+ = \emptyset$ or $A_- = \emptyset$. By a reversal of time we may assume that $A_+ = \tilde{M}$.

For any continuous map $\theta: M \to S^1$, let $F_1: M \times \mathbf{R} \to S^1$ be defined by $F_1(x, t) = \theta(f_t(x))$ and let $F_2(x, t) = \theta(x)$. Then, since F_1 is homotopic to F_2, it follows that the map $\Delta_1\theta: M \times \mathbf{R} \to S^1$ defined by $\Delta_1\theta = F_1F_2^{-1}$ (where we use the natural group structure in the set of maps from $M \times \mathbf{R}$ into S^1) is homotopic to a constant. Let $\Delta\theta: M \times \mathbf{R} \to \mathbf{R}$ be such that $\exp\circ\Delta\theta = \Delta_1\theta$. Then, $\Delta\theta(x, t)$ measures the net change in argument of the angular function θ, along the piece of trajectory going from x to $f_t(x)$. If θ is such that $\Delta\theta$ is strictly increasing along trajectories, then, as was observed by Birkhoff [3], $\theta^{-1}(1)$ is a (topological) cross-section for the flow.

From above, it follows that for each $\tilde{x} \in \tilde{M}$, $\lim_{t\to\infty} \tilde{g}(\tilde{f}_t(\tilde{x})) = \infty$ and it is very easy to see that this implies that $\lim_{t\to\infty} \Delta g(x, t) = \infty$.

Hence, by Theorem 1 in Fuller [10] it follows that there exists a continuous $G: M \to S^1$, homotopic to g, and such that for each $x \in M$ the map $t \to \Delta G(x, t)$ is a strictly increasing function of t. Thus G is an angular function such that the argument is strictly increasing along trajectories of f_t. Using flow boxes and the fact that ΔG increases along trajectories, one verifies easily that $G^{-1}(1)$ is a compact, flat submanifold of M. In fact, there exists a homeomorphism $H: [-1, 1] \times G^{-1}(1) \to V$, where V is a neighborhood of $G^{-1}(1)$, such that, for each $x \in G^{-1}(1)$, $H(0, x) = x$ and $H([-1, 1] \times \{x\})$ is a connected piece of trajectory of the flow. Since $G^{-1}(1)$ is bicollared by a smooth flow it follows by [37], that given $\epsilon > 0$ there exists a continuous isotopy $h_s: M \to M$, $0 \leq s \leq 1$, such that $d(h_s(x), x) \leq \epsilon$, for all $x \in M$ and $s \in I$; and $h_1(G^{-1}(1))$ is a smooth submanifold of M which is transversal to the flow. Hence, f_t admits a smooth global cross-section, and by the remarks preceding the theorem we have proved sufficiency.

Remark 4.2. Theorem 4.1 can be rephrased as follows:

A codimension one Anosov flow $f_t: M \to M$ is topologically equivalent to the suspension of a hyperbolic toral isomorphism if and only if $H^1(M, \mathbf{Z}) = \mathbf{Z}$ and for every periodic orbit γ, the inclusion $i: \gamma \to M$ induces a monomorphism $i^*: H^1(M, \mathbf{Z}) \to H^1(\gamma, \mathbf{Z})$. Furthermore, if $H^1(M, \mathbf{Z}) = \mathbf{Z}$ and if every periodic orbit is not cohomologically trivial (in the above sense) and if $g: M \to S^1$ is such that its homotopy class $[g] \in [M, S^1]$, generates $H^1(M, \mathbf{Z})$ (here we use the isomorphism $[M, S^1] \approx H^1(M, \mathbf{Z})$ of Eilenberg-MacLane) then, there exists a differentiable map $G: M \to S^1$, homotopic to g, and such that G is a locally trivial submersion, and $G^{-1}(1)$ is a cross-section for f_t.

Remark 4.2. Using exactly the same arguments in the proof of the sufficiency part of Theorem 4.1, we can prove the following theorem:

Let $f_t: M \to M$ be an Anosov flow on the compact, connected, smooth manifold M (with f_t not necessarily of codimension one). If $\Omega(f_t) = M$, and if there exists an epimorphism $\varphi_*: H_1(M, \mathbf{Z}) \to \mathbf{Z}$ such that for each periodic orbit $\varphi_*([\gamma]) \neq 0$ (where $[\gamma]$ denotes the homology class of γ), then f_t admits a cross-section, obtained as $\varphi^{-1}(1)$ where $\varphi: M \to S^1$ is a locally trivial submersion having as induced homomorphism, in first integral homology groups, φ_*. Hence f_t is topologically equivalent to the suspension of an Anosov diffeomorphism.

Remark 4.3. Theorem 4.1 is a partial answer to a conjecture given by Plante in [26].

An Example. Let M^3 be the unit tangent bundle over a compact riemannian surface with constant negative curvature. Let $g_t: M^3 \to M^3$ be its geodesic flow and let $TM = E^s \oplus E^u \oplus E^1$ be its Anosov splitting into a Whitney sum. Then, as is well known, E^s, E^u and E^1 are real analytic line bundles. Then $E^s \oplus E^1$ gives an analytical foliation. Let the 2-dimensional distribution $E^s \oplus E^1$ be given by the map $\Gamma: M^3 \to G_2(M^3)$ where $G_2(M^3)$ denotes the Grassmanian bundle of planes over M^3. Let $C^1(M^3, G_2(M^3))$ denote the space of sections with the C^1 topology. We claim that there exists $\epsilon > 0$ such that if $\bar{\Gamma}$ is any integrable distribution contained in the ϵ-neighborhood of Γ, in $C^1(M^3, G_2(M^3))$, then $\bar{\Gamma}$ does not contain a compact leaf.

Proof. Let X be the vector field of g_t and let Y be a nonsingular, smooth, vector field which generates the line bundle E^u. Let $\delta > 0$ be sufficiently small so that $\bar{X} = X + \delta Y$ is an Anosov vector field. Let $\epsilon > 0$ be sufficiently small so that any integrable distribution $\bar{\Gamma}$ in the ϵ-neighborhood of Γ is transversal to $\bar{X}$ and the leaves of $\bar{\Gamma}$ meet every periodic orbit of the flow generated by $\bar{X}$ (the latter is possible since the leaves of Γ are dense in M). Then, every such $\bar{\Gamma}$ does not have a compact leaf. Otherwise, if $\bar{\Gamma}$ had a compact leaf, Σ, this compact leaf would meet, transversally, every orbit of the codimension one Anosov flow generated by $\bar{X}$. This would be true because Σ meets every periodic orbit and by a theorem of Newhouse [23] if a flow on a compact manifold satisfies Axiom A', then every orbit contains a periodic orbit on its closure. Hence, Σ would be a cross-section for the flow generated by $\bar{X}$ and, therefore, Σ would be a 2-torus and M^3 would be a torus bundle over S^1. This would be a contradiction because one can prove that a 3-manifold which is a suspension manifold of a hyperbolic linear isomorphism $A: T^2 \to T^2$ has trivial centre in its fundamental group, and, in our case, this centre is free cyclic.

Remark 4.4. M. Hirsch has very general results of this type in [14].

CENTRO DE INVESTIGACIÓN DEL IPN

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