JACOBIANS OF CURVES WITH g_4^1'S ARE THE PRYM'S OF TRIGONAL CURVES

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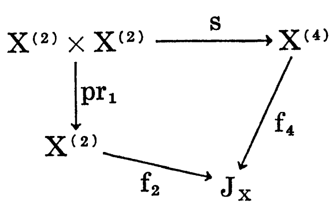
In this note we show that the Prym varieties of trigonal curves are always Jacobians of curves. I thank G. Kempf for the useful conversation on the topic.

Curve will always mean a complete non-singular irreducible curve defined over $\mathbf{C}$, $\mathbf{P}^1 = \mathbf{P}_{\mathbf{C}}^1$.

Let X be a non-hyperelliptic curve of genus g_X which has a g_4^1; we will assume that this linear system does not contain divisors of the form $2P + 2Q$ or $4P$.

Let J_X be the Jacobian variety of X, $X^{(n)}$ the n^{th} symmetric product of X, $f\colon X^{(n)} \to J_X$ the morphism $D \to$ div. class $(D - nP_0)$, $P_0 \in X$ fixed, $W^n = Im\, f_n$ and $s\colon X^{(n)} \times X^{(n)} \to X^{(2n)}$ the sum map.

Consider now the diagram:



If $|D|$ is our g_4^1, then consider the curve $\tilde{C} = s^{-1}(|D|)$. This curve is non-singular and of genus $2g_X + 1$ (Proposition 1), moreover the involution $(d, d') \to (d', d)$ of $X^{(2)} \times X^{(2)}$ induces a fixed point free involution $*\colon \tilde{C} \to \tilde{C}$ such that $s \circ * = s$, that is $C = \tilde{C}/*$ is a smooth curve of genus $g_X + 1$, which has a morphism $s'\colon C \to |D|$ of degree 3 i.e. C is a trigonal curve. So we have the following:

Theorem. *Let X be a non-hyperelliptic curve which has a g_4^1 which does not contain divisors of the form $2P + 2Q$ or $4Q$. Let $\tilde{C}$ and C be constructed as before; then J_X is naturally isomorphic to the Prym variety of $\tilde{C} \to C$. Moreover all Prym varieties of trigonal curves arise in this way.*

To prove the first part we use the following criterion ([M]):

Let $\tilde{C}$ be a curve which has a fixed point free involution $*$. If there is a principally polarized abelian variety (A, Σ) and a morphism $\phi\colon \tilde{C} \to A$ such that 1) $\phi \circ * = \phi$ and 2) $\phi(\tilde{C}) \equiv 2\Sigma^{(g-2)}/(g-2)!$ (num. equivalence), g = genus of $\tilde{C}/*$. Then ϕ induces an isomorphism $P \to A$ (as principally polarized abelian varieties) where P is the Prym variety of $\tilde{C} \to C$.

To check 1), as before let $\phi = f_2 \circ pr_1 - d/2$ where $d = f_4(|D|)$. Then $\phi(\tilde{C}) = (W_2 - d/2)\cdot(d/2 - W_2)$, so clearly $\phi \circ * = \phi$.

To check 2)† let $a\colon X \to \mathbf{P}^1$ be a good g_4^1, let Δ denote a diagonal and $V = \{(x, y) \mid a(x) = a(y)$ with $x \neq y$ generically$\}$

† This computation is due to G. Kempf.

So

$$(a \times a)^{-1}\Delta_{\mathbf{P}^1} = \Delta_X + V.$$

On the other hand

$$\Delta_{\mathbf{P}^1} \underset{rat}{\simeq} \mathbf{P}^1 \times \{p\} + \{q\} \times \mathbf{P}^1 \qquad p, q \in \mathbf{P}^1$$

So

$$(a \times a)^{-1}\Delta_{\mathbf{P}^1} \underset{rat}{\simeq} X \times D_1 + D_2 \times X \qquad D_1, D_2 \in |D|.$$

The image of this relation in $X^{(2)}$ under the sum map is:

$$s_*(\Delta_X) + s_*(V) \underset{rat}{\simeq} s_*(X \times D_1) - s_*(D_2 \times X)$$

Since Δ_X and $X \times \{p\}$ go isomorphically into $X^{(2)}$ and V is a double covering of $\tilde{C}$ in $X^{(2)}$ we have:

$$\delta + 2\tilde{C} \underset{rat}{\simeq} 2(\Sigma_j X + p_j); \quad p_1 + \cdots + p_4 \in |D|$$

where δ and $X + p_j$ are the images of Δ_X and $X \times p_j$ respectively. If $f = f_2$ and $\mathbf{X}$ is the canonical image of X in J_X then:

$$f_*(\delta) + 2f_*(C) \underset{rat}{\simeq} 2\Sigma_j f_*(X + p_j).$$

One has also $f_*(\delta) = (2\, id_{J_X})\mathbf{X} \underset{num}{\equiv} 4\mathbf{X}$

$$\text{and} \quad 2(\Sigma_j f_*(X + p_j)) \underset{alg}{\sim} 8\mathbf{X}$$

Hence:

$$f_*(\tilde{C}) \underset{num}{\equiv} 2\mathbf{X}$$

We will prove the second part of the theorem by showing that the previous construction of an unramified double cover of a trigonal curve $\tilde{C} \to C$ from a given curve with a good g_4^1 is the same as the one given by a morphism between Hurwitz spaces, and in the later case we show that every unramified double cover of a trigonal curve comes from a curve with a good g_4^1 (Proposition 2).

Let X be a curve, $a\colon X \to \mathbf{P}^1$ a finite morphism of degree n. Then a is an n-branched cover from a compact Riemann surface onto the Riemann sphere. Denote by $\delta(a)$ the branch locus of a; so if $X_0 = X - a^{-1}(\delta(a))$ and $a_0 = a|_{X_0}$, then $a_0 : X_0 \to \mathbf{P}^1 - \delta(a)$ is a covering space of degree n (connected). Given $x \in \mathbf{P}^1 - \delta(a)$, the action of $\pi_1(\mathbf{P}^1 - \delta(a), x)$ on $a^{-1}(x)$ induces an homomorphism $\bar{a}\colon \pi_1(\mathbf{P}^1 - \delta(a), x) \to S_n$, where S_n is the n-th symmetric group.

It is classically known that the above establishes a 1–1 correspondence between the classes of objects considered (isomorphism classes in the case of maps and classes modulo inner automorphisms in the case of group homomorphisms). We will denote by (a) or $(\bar{a})$ corresponding classes.

Let $a\colon X \to \mathbf{P}^1$ be given by a g_4^1. So $a_0\colon X_0 \to \mathbf{P}^1 - \delta(a)$ is a covering space of degree 4. Also if $s\colon \tilde{C} \to \mathbf{P}^1$ and $s'\colon C \to \mathbf{P}^1$ are as before, let $s_0 : C_0 \to \mathbf{P}^1 - \delta(a)$ and $s'_0 : C_0 \to \mathbf{P}^1 - \delta(a)$ be corresponding covering spaces (of degree 6 and 3 respectively).

Let $k\colon S_4 \to S_3$ be the homorphism whose kernel is the Klein group and $b\colon S_4 \to S_6$ be the faithful representation of S_4 given by its action on the left cosets of the subgroup $F = \{1, t, t', tt'\}$ ($t, t' \in S_4$ transpositions such that $tt' = t't$). Then we have the following

Remark: Let $\bar{a}\colon \pi_1(\mathbf{P}^1 - \delta(a), x) \to S_4$ be an homomorphism whose class corresponds to the class of a_0. Then $b\circ\bar{a}\colon \pi_1(\mathbf{P}^1 - \delta(a), x) \to S_6$ is an homomorphism whose class corresponds to the class of s_0 and $k\circ\bar{a}\colon \pi_1(\mathbf{P}^1 - \delta(a), x) \to S_3$ is an homomorphism whose class corresponds to the class of s_0'.

Proof: The first statement follows from the fact that the fibers of $s_0^{-1}(p)$ of s_0 can be identified with the set of subsets of $a_0^{-1}(p)$ of cardinality two. Moreover with these identifications, the involution $*$ corresponds to the fact that $[k^{-1}(k(F))\colon F] = 2$, so the class of $k\circ\bar{a}$ corresponds to the class of s_0'.

Proposition 1: *If $a\colon X \to \mathbf{P}^1$ is given by a g_4^1 which does not contain any divisor of the form $2P + 2Q$* or *$4P$, then $s\colon \tilde{C} \to \mathbf{P}^1$ is a branched cover from a compact Riemann surface of genus $2g_X + 1$. Moreover since $*\colon \tilde{C} \to \tilde{C}$ is fixed point free, $s'\colon C \to \mathbf{P}^1$ is a branched cover of degree 3 from a compact Riemann surface C of genus $g_X + 1$, that is to say C is trigonal.*

Proof: The assumption on the linear system means that above each branch point $p \in \delta(a)$ there is only one ramification point, with ramification index 2 or 3. So if σ is an element of $\pi_1(\mathbf{P}^1 - \delta(a), x)$ which "goes once around p", then $\bar{a}(\sigma)$ is either a transposition or a cycle of order three. By looking at an explicit form of the homomorphism $b\colon S_4 \to S_6$, one can see that if $\bar{a}(\sigma)$ is a transposition (resp: a cycle of order three) on S_4, then $b(\bar{a}(\sigma))$ is the product of two transpositions (resp: of two cycles of order three) which commute. This means that on $s\colon \tilde{C} \to \mathbf{P}^1$ above p there are two ramification points, each of index two (resp: three). So s is a branched cover with a total ramification index twice that of a. Thus by the Riemann-Hurwitz formula, $g_X = -3 + w$ and $g_{\tilde{C}} = -5 + 2w$, hence $2g_X + 1 = g_{\tilde{C}}$.

The second part of the proposition follows from the first.

Let Σ^s be the complex manifold consisting of all unordered s-tuples of points in $\mathbf{P}^1$, $H(n, s)$ the set of isomorphism classes of n-branched covers of $\mathbf{P}^1$ with s-branch points and $\delta\colon H(n, s) \to \Sigma^s$ the branch morphism.

One can give a topology to $H(n, s)$ (due to Hurwitz) so that δ becomes a covering space and in this way $H(n, s)$ inherits the complex structure of Σ^s.

Let $N(U_1, \cdots, U_s)$ be the subset of Σ^s consisting of the s-tuples of points having one point on each U_i, where the U_i are disjoint open disks in $\mathbf{P}^1$. Such sets form a basis for the topology of Σ^s. For any $A, A' \in N(U_1, \cdots, U_s)$, since $\mathbf{P}^1 - U$ is a deformation retract of $\mathbf{P}^1 - A$ and $\mathbf{P}^1 - A'$, where $U = \bigcup_{i=1}^{s} U_i$, we have isomorphisms $\phi_{A,A'}\colon \pi_1(\mathbf{P}^1 - A, x) \to \pi_1(\mathbf{P}^1 - U, x) \to \pi_1(\mathbf{P}^1 - A', x)$ which do not depend on $U_1, \cdots, U_s$.

So given $(f) \in H(n,s)$ and a neighborhood $N(U_1, \cdots, U_s)$ of (f), the neighborhood of (f) above this one will be:

$$N(U_1, \cdots, U_s)_{(f)} = \{(f \circ \phi_{A,\delta(f)}) \mid A \in N(U_1, \cdots, U_s)\}.$$

From Proposition 1 it follows that if $(a) \in H(4,s)$ is given by a good g_4^1, then $(b \circ \bar{a}) \in H(6,s)$ and $(k \circ \bar{a}) \in H(3,s)$. This defines Σ-maps B: $U \rightarrow H(6,s)$ and K: $U \rightarrow H(3,s)$ where U is the open set (not necessarily connected) of all branched covers which have above each branch point only one ramification point and of index at most three.

We list some properties of these maps:

i) They are analytic. This follows from the definition of the analytic structure of the $H(n,s)$'s.

ii) They commute with the action of $PGL(1)$, where such action is given by $(\bar{a}) \rightarrow (\lambda \circ \bar{a})$ for $\lambda \in PGL(1)$.

iii) They are algebraic. This follows from a result that appears in Grothendieck [G], since Σ is algebraic and δ is a covering space of finite degree.

The remark shows that the construction of an unramified double cover of a trigonal curve from a given curve with a good g_4^1, is the same as the one given by the Σ-morphisms B and K. So the last part of the theorem will follow from:

PROPOSITION 2: 1) *B is injective and K is surjective. Moreover the cardinality of a fiber $K^{-1}(e)$ is equal to $2^{w-4} - 1$, where w is the total ramification index of e.*
2) *If $e: C \rightarrow \mathbf{P}^1$ is a branched cover of degree 3, the fiber $(K \circ B^{-1})^{-1}(e)$ can be identified in a natural way with the set of the unramified double covers of C.*

This proposition is similar to the one we proved in an earlier paper [R], except that here we do not restrict ourselves to simple covers. It is proved in the same way.

Injectivity of B follows from the injectivity of b and from the fact that S_4 does not have any exterior automorphisms.

Let e: $C \rightarrow \mathbf{P}^1$ be a branched cover of degree three with s branch points and total ramification index ω.

From $[k^{-1}(k(F)):F] = 2$ follows that $(K \circ B^{-1})^{-1}(e)$ can be identified with a subset of the unramified double covers of C. So both sets are equal if we show that Card $(K^{-1}(e)) = 2^{\omega-4} - 1$ since $\omega - 4 = 2g_C$.

To do this assume that we have a basis $\sigma_1, \cdots, \sigma_s$ of π_1 where $\sigma_1 \cdots \sigma_s = 1$ and such that e corresponds to $\bar{e}: \pi_1 \rightarrow S_3$ where $\bar{e}(\sigma_1), \cdots, \bar{e}(\sigma_r)$ are cycles of order 3 and $\bar{e}(\sigma_{r-1}), \cdots, \bar{e}(\sigma_s)$ are transpositions ($0 \leq r \leq s$, $s - r$ even and $\omega = 2r + s - r$).

So we want to compute how many non-equivalent (up to inner automorphism) s-sequences $\{P_1, \cdots, P_s\}$ of elements of S_4 one can construct such that: a) $k(P_1) = \bar{e}(\sigma_i)$; b) $P_1 \cdots P_s = 1$ and c) $P_1, \cdots, P_s$ generate a transitive subgroup of S_4.

Since for $1 \leq i \leq r$ we have that $k^{-1}(\bar{e}(\sigma_i))$ consists of 4 cycles of order 3 and for $r < i \leq s$, $k^{-1}(\bar{e}(\sigma_i))$ contains 2 transpositions it follows that:

If $r = s$, for any choice of $P_1, \cdots, P_{s-1}$; P_s is uniquely determined by $P_1 \cdots P_s = 1$. Thus we have $4^{s-1} = 2^{\omega-2}$ choices.

If $r < s$, for any choice $P_1, \cdots, P_{s-2}$; P_{s-1} and P_s are uniquely determined since they have to be transpositions and such that $P_1 \cdots P_s = 1$. So we have $4^r \cdot 2^{r-s-2} = 2^{\omega-2}$ choices. From the number $2^{\omega-2}$ we have to substract 4, since this is the number of sequences $P_1, \cdots, P_s$ which satisfying a) and *b*) generate an intransitive subgroup of S_4. We now have to divide by 4, since they are three inner automorphisms of S_4 which become trivial in S_3, one being the composition of the other two. So we get $2^{\omega-4} - 1$ non equivalent sequences $P_1, \cdots, P_s$ which satisfy a), b) and c).

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