UNITARY APPROXIMATIONS TO FRAMED BORDISM

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Introduction.

The purpose of this note is to exhibit framed bordism homology theory as the inverse limit of other bordism theories; these approximating theories are associated with unitary bordism. Let $k\xi_n$ be the k-fold Whitney sum of the canonical unitary bundle over $BU(n)$ and let $M(k\xi_n)$ be the associated Thom space. With k fixed, $M(k\xi_n)$ can be considered the kn^{th} space of a spectrum, which we denote MkU. We denote the homology or cohomology defined by these spectra by the same symbols. Clearly $M1U$ is just unitary bordism MU. If k' divides k, there is a canonical map $MkU \to Mk'U$ defined as follows: let $k = \ell k'$ and consider $\phi_\ell \colon BU(n) \to BU(\ell n)$ which is

$$BU(n) \xrightarrow{\Delta} \underset{\ell\text{-fold}}{\times} BU(n) \xrightarrow{\oplus} BU(\ell n).$$

Then $\phi_\ell^*(k'\xi_{\ell n}) = k\xi_n$ and hence there is an induced map $\phi_\ell \colon MkU \to Mk'U$. We consider the collection of MkU as an inverse system of spectra. Note that there is a canonical map of the sphere spectrum S to each MkU as the multiplicative unit.

Framed bordism has been denoted by a variety of symbols, such as π_*^s (classical), Ω_*^1 (Buhstaber-Novikov), Ω_*^{Fr} (Conner-Smith), and S_* (Adams-Ray). Not to be left out, we denote framed bordism by $M1_*$, to emphasize it comes from the Thom spectrum of the trivial group. We will continue to let S denote the sphere spectrum, however. Let X be a space with finitely generated integral homology groups.

The main result of this note is the following.

1.1 Theorem. *The natural map $M1_*(X) \to \operatorname{inv\,lim} MkU_*(X)$ is an isomorphism.*

If X is a finite complex, the same formula holds in cohomology by Spanier-Whitehead duality. It is definitely not true for all spaces; for example, if X is a rational space the result is false. Neither is it the case that the sphere spectrum is the inverse limit of the spectra MkU. The same result is true at each prime; however it seems difficult to prove this directly. Let $Z_{(p)}$ denote the integers localized at the prime p.

1.2 Corollary. *For each prime p, there is a natural isomorphism $M1_*(X) \otimes Z_{(p)} \to \operatorname{inv\,lim}\, (MkU_*(X) \otimes Z_{(p)}) = \operatorname{inv\,lim}\, (Mp^rU_*(X) \otimes Z_{(p)})$.*

There have been other expressions of $M1_*$ as an inverse limit of homologies, say $Y_{\gamma *}$. However, in these cases the cones of the maps $S \to Y_\gamma$ have become more and more connected, so the result is trivial. Such is not the case here, since for example, by the Thom isomorphism, $H_*(MkU)$ is a polynomial algebra on even dimensional generators for each k. Indeed each Φ_ℓ induces an isomorphism on rational homology or homotopy.

As it stands (1.1) is nothing more than a curiosity, for it tells us nothing about $M1_*$. However, the MkU are pleasant in several ways and it may be easier to compute in the MkU_* rather than directly in $M1_*$. For example, $\pi_*(MkU)$ has a large torsion-free part. Hence, elements in $\pi_*(S)$ that are indecomposable (even as Massey products) may become decomposable (perhaps as Massey products) in each of the $\pi_*(MkU)$. This would be reflected in any algebraic machine used to compute; e.g., the Adams spectral sequence. Some of this algebraic machinery has been developed in [4]. Also, it seems possible the $\{\pi_*(MkU)\}$ can be characterized algebraically, perhaps in terms of formal groups. From this, perhaps $\pi_*(S)$ can be given a purely algebraic characterization.

The elements of $\pi_*(MkU)$ are cobordism classes of triples (M, η, ψ) where M is a differentiable manifold embedded in Euclidean space with normal bundle ν say, η is a complex bundle on M, and ψ is a stable equivalence between $k\eta$ and ν. That is ν admits a stable reduction of its group to the k-fold direct sum of the standard representation of the unitary group. If $k = 2$, the second copy of η can be conjugated to give M a stable symplectic structure. Thus we have a canonical map $M2U \to MSp$. Such manifolds were mentioned in [3].

2. Recollections

We consider [1] a convenient general reference for facts about generalized homologies, derived limit functors, etc. In particular, recall that $MkU_*(X)$ is defined as $\pi_*(MkU \wedge X)$. Concerning inverse limits: all of our index sets I for projective systems will be countable, and all of our groups will be commutative. If $\{G_i\}_{i\in I}$ is a projective system of finite groups,

$$\lim{}^1 G_i = 0 \tag{2.1}$$

by the Mittag-Leffler condition. If

$$0 \to A_i \to B_i \to C_i \to 0$$

is a projective system of exact sequences of groups, the basic exact sequence

$$\begin{aligned} 0 \to \lim{}^0 A_i \to \lim{}^0 B_i \to \lim{}^0 C_i \to \lim{}^1 A_i \\ \to \lim{}^1 B_i \to \lim{}^1 C_i \to 0 \end{aligned} \tag{2.2}$$

is obtained. Also we remark that the limit functors defined by I', a cofinal subset of I, are the same as those defined by I.

For any group G, let $Tor\ G$ be the torsion subgroup of G, and $Free\ G = G/Tor\ G$ the torsion-free quotient of G.

3. The proof

Let $\beta_i \in H_{2i}(BU)$, $i = 0, 1, 2, \cdots$ $(\beta_0 = 1)$ be the canonical polynomial generators of $H_*(BU)$ and let $b_i(k) \in H_{2i}(MkU)$ be their counterparts under the Thom isomorphism in $H_*(MkU)$. Let $b(k) = 1 + b_1(k) + b_2(k) + \cdots$. Recall there is a map $\phi_\ell: M\ell kU \to MkU$. It is elementary to show

$$(\phi_\ell)_* b(\ell k) = (b(k))^\ell$$

and in particular, if $i > 0$,

$$(\phi_\ell)_* b_i(\ell k) \equiv \ell b_i(k) \text{ modulo decomposables.}$$

Order the monomials in the b_j of degree i by the sum of their exponents, and extend to a linear ordering. With respect to this basis, the matrix of $(\phi_\ell)_*$: $H_{2i}(M\ell kU) \to H_{2i}(MkU)$ is triangular with all diagonal entries divisible by ℓ. Focus attention on a prime p, and consider $(\phi_{p^s})_* = (\phi_p)_*^s$. If s is large enough (depending on i), the matrix of this map will be zero modulo p. That is:

(3.1) *for each integer $i > 0$ and prime p, there exists an integer $s = s_p(i)$ such that if p^s divides ℓ, all elements in*

$$\text{image } ((\phi_\ell)_*: H_{2i}(M\ell kU) \to H_{2i}(MkU))$$

are divisible by p,

(3.2) *for each k*

$$\bigcap_\ell \text{ (image: } H_{2i}(M\ell kU) \to H_{2i}(MkU)) = \{0\}$$

if $i > 0$.

The index set for the projective system $\{MkU\}$ is the positive integers, ordered by divisibility. Choose and fix a cofinal subset I isomorphic to the positive integers with the linear ordering. Also fix a connected space X with finitely generated homology groups. In the homotopy category, according to [2], there exists an inverse limit Y of the system of spectra $\{MkU \wedge X\}_{k \in I}$ and a natural commutative diagram of Milnor exact sequences

$$\begin{array}{ccccccccc} 0 \to & \lim^1 \pi_*(MkU \wedge X) & \to & \pi_*(Y) & \to & \lim^0 \pi_*(MkU \wedge X) & \to 0 \\ & \downarrow h_1 & & \downarrow h_Y & & \downarrow h_0 & \\ 0 \to & \lim^1 H_*(MkU \wedge X) & \to & H_*(Y) & \to & \lim^0 H_*(MkU \wedge X) & \to 0 \end{array} \tag{3.3}$$

The vertical maps h are induced by the Hurewicz homomorphism.

We claim:

(3.4) $\lim^1 H_*(MkU \wedge X)$ *is torsion-free and divisible*; *hence is a rational vector space*,

(3.5) h_1 *is an isomorphism.*

To prove (3.4), note that

$$H_j(MkU \wedge X) = \sum_{2i+r=j} H_{2i}(MkU) \otimes H_r(X)$$

Fix a j and a prime p. Choose a cofinal subset of I, say $I' = \{k_1 < k_2 < \cdots\}$ so that each k_i divides k_{i+1} by p^s where

$$s = \max \{s_p(i) \mid 0 < i \leq \tfrac{1}{2} j\},$$

the $s_p(i)$ coming from (3.1). Note that

$$\lim{}^1 (H_*(MkU) \otimes H_*(X)) = \lim{}^1 (\tilde{H}_*(MkU) \otimes H_*(X)),$$

where

$$\tilde{H}_*(MkU) = \textstyle\sum_{i>0} H_{2i}(MkU).$$

By (3.1), each element in the image of

$$\theta_m = (\phi_\ell)_*: \tilde{H}_*(Mk_{m+1}U) \otimes H_*(X) \rightarrow \tilde{H}_*(Mk_mU) \otimes H_*(X)$$

is divisible by p ($\ell = k_{m+1}/k_m$).

Let $\Pi_j = \Pi_m(\tilde{H}_*(Mk_mU) \otimes H_*(X))_j$. Let $<x_m> = <x_{k_1}, x_{k_2}, \cdots >$ be the elements of Π_j. Consider $F: \Pi_j \rightarrow \Pi_j$ defined by $F < x_m > = < x_m - \theta_m x_{m+1} >$. By definition, the sequence

$$0 \rightarrow \lim{}^0 (\tilde{H}_*(MkU) \otimes H_*(X))_j \rightarrow \Pi_j \xrightarrow{F} \Pi_j$$
$$\rightarrow \lim{}^1 (\tilde{H}_*(MkU) \otimes H_*(X))_j \rightarrow 0$$

is exact. Call the last term $\lim^1{}_j$ for short. Given $<x_m> \in \Pi_j$, let $<x_m>^{\mathrm{I}}$ denote its class in $\lim^1{}_j$.

Suppose $p<x_m>_1 = 0$; i.e. $p<x_m> = <y_m - \theta_m y_{m+1}>$ in Π_j. Let $\theta_m y_{m+1} = pz_m$; then $<x_m> = <z_m - \theta_m z_{m+1}>$, and $<x_m>_1 = 0$. That is, $\lim^1{}_j$ has no p-torsion.

Now suppose given $<x_m> \in \Pi_j$. Let $\theta_m x_{m+1} = py_m$. Then $<x_m> = p<y_m> + <x_m - \theta_m x_{m+1}>$ in Π_j, so $<x_m>_1 = p<y_m>_1$. That is $\lim^1{}_j$ is p-divisible. Since p and $j > 0$ were arbitrary, (3.4) is proved.

We turn to the proof of (3.5). If $H_*(X)$ is finitely generated in each dimension, so is $MkU_*(X) = \pi_*(MkU \wedge X)$ by the Atiyah-Hirzebruch spectral sequence. In particular, $Tor\ \pi_*(MkU \wedge X)$ is finite in each dimension, as is $Tor\ H_*(MkU \wedge X)$. Thus by (2.1) and (2.2), we have a commutative diagram

$$\begin{array}{ccc}
\lim^1 \pi_*(MkU \wedge X) & \xrightarrow{\approx} & \lim^1 Free\ \pi_*(MkU \wedge X) \\
\Big\downarrow h_1 & & \Big\downarrow h' \\
\lim^1 H_*(MkU \wedge X) & \xrightarrow{\approx} & \lim^1 Free\ H_*(MkU \wedge X)
\end{array}$$

The stable Hurewicz homomorphism is a rational equivalence, so h' is a monomorphism with finite cokernel in each dimension. Let K_k denote the graded group of cokernels. The sequence

$$0 \rightarrow \lim{}^0 Free\ \pi_*(MkU \wedge X) \rightarrow \lim{}^0 Free\ H_*(MkU \wedge X) \xrightarrow{\alpha} \lim{}^0 K_k$$
$$\rightarrow \lim{}^1\ Free\ \pi_*(MkU \wedge X) \xrightarrow{h'} \lim{}^1\ Free\ H_*(MkU \wedge X)$$
$$\rightarrow \lim{}^1\ K_k \rightarrow 0$$

is exact by (2.2). By (2.1), $\lim^1 K_k = 0$. For each k, the Hurewicz homomorphism breaks up as follows:

$$\begin{array}{ccccc}
\pi_*(X) & \longrightarrow & \pi_*(MkU \wedge X) & & \\
\big\downarrow h_X & & \big\downarrow h & \searrow \tilde{h} & \\
H_*(X) & \longrightarrow & H_*(X) \oplus \tilde{H}_*(MkU) \otimes H_*(X) & \longrightarrow & \tilde{H}_*(MkU) \otimes H_*(X) \longrightarrow 0
\end{array}$$

The map h_X is the stable Hurewicz homomorphism of X; hence its cokernel is independent of k. By (3.1), the inverse limit of the cokernel of $\tilde{h}$ is zero. Hence α is surjective, h' is an isomorphism, and (3.5) is proved.

Now let $f\colon S \wedge X \to Y$ be the natural map and let Cf be the cone of f. Since $\lim^0 H_*(MkU \wedge X) = H_*(X)$ by (3.2), the homology sequence of the cofibration

$$(3.6) \qquad S \wedge X \to Y \to Cf$$

is a splitting of the bottom row of (3.3). Hence Cf is a rational spectrum by (3.4) and its Hurewicz homomorphism is an isomorphism. By (3.5), the top row of (3.3) is a splitting of the homotopy sequence of (3.6). Therefore

$$M1_*(X) = \pi_*(S \wedge X) \xrightarrow{\approx} \lim^0 \pi_*(MkU \wedge X) = \lim^0 MkU_*(X)$$

and (1.1) is proved.

(3.7) *Remark.* Note that, as a corollary, a splitting of the spectrum Y has been produced: $Y = (S \wedge X) \vee Cf$.

We turn now to corollary 1.2. It is trivial once it is established that

$$(3.8) \qquad \lim^0 (MkU_*(X) \otimes Z_{(p)}) = (\lim^0 MkU_*(X)) \otimes Z_{(p)}$$

However, once we know that $\{MkU_*(X)\}$ converges to finitely generated groups' it is fairly routine to establish (3.8). We leave the details to the reader.

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