ON EQUIVARIANT MAPS OF LOW CODIMENSION FROM REAL PROJECTIVE SPACES TO SPHERES

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0. Introduction

Let $P^{(n)}$ be real projective n-space P^n if n is odd, and the union of two real projective n-spaces $P_1{}^n \cup P_2{}^n$ with $P_1{}^n \cap P_2{}^n = P^{n-1}$ if n is even. $P^{(n)}$ admits a fixed point free involution, as does the n-sphere (the antipodal map). The *coindex* of $P^{(n)}$ is the least integer k for which there exists an equivariant map $P^{(n)} \to S^k$. We study the set of all elements of $\pi_{n+k}S^n$ which admit representatives of the form

$$S^{n+k} \subset S^{(n+k)} \xrightarrow{\pi} P^{(n+k)} \xrightarrow{f} S^n$$

where f is equivariant (see §2 for the definition of $S^{(n+k)}$). For low values of k (≤ 3) our results are nearly complete. Application is made to give new information on coindex $P^{(n)}$ for $n \leq 16$.

Notations and unreferenced results concerning the homotopy groups of spheres can be found in Toda's book [10].

1. Equivariant elements of $\pi_3 S^2$

The standard Z_m-action (S^{2k-1}, λ_m) is given by $\lambda_m(z_1, \cdots, z_k) = (e(1/m)z_1, \cdots, e(1/m)z_k)$, where $e(x) = \exp(2\pi ix)$. For definitions and preliminaries of equivariant maps, we refer the reader to [6, §2]. In particular, we will make use of

Proposition 1.1 (Folkman [6]) (i) *Let (X, T_2) be a Z_m-action with X path connected, $(2k-1)$-simple and $T_2 \sim id$. Then for any map $f:(S^{2k-1}, \lambda_m) \to (X, T_2)$ and any $\alpha \in \pi_{2k-1}X$, there exists a map $g: (S^{2k-1}, \lambda_m) \to (X, T_2)$ such that $[g] = [f] + m\alpha$.*

(ii) *Let (X, T_2) be a Z_m-action with X path connected, n-simple for $n = 1, 2, \cdots, 2k-1$ and $T_2 \sim id$. Suppose* Hom $(Z_m, \pi_{2i-1}X) =$ Ext $(Z_m, \pi_{2i}X) = 0$ *for $i = 1, 2, \cdots, k-1$. If $f, g: (S^{2k-1}, \lambda_m) \to (X, T_2)$ are maps, then $[g] - [f] = m\alpha$ for some $\alpha \in \pi_{2k-1}X$.*

Folkman's proofs of (i) and (ii) are valid with the weaker hypothesis "$(T_2)_\# : \pi_{2k-1}X \to \pi_{2k-1}X$ is the identity isomorphism" replacing the hypothesis "$T_2 \sim id$". Furthermore, if the hypothesis $T_2 \sim id$ in 1.1 (i) is eliminated, then the proof in [6] establishes the weaker conclusion $[g] - [f] = \sum_{j=0}^{3} (T_2{}^j)_\# \alpha$.

Lemma 1.2 (i) *There exists a map $f: (S^{2k-1}, \lambda_m) \to (S^{2k-1}, \lambda_m)$ of degree d if and only if $d = mj + 1$ for some j.*

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(ii) *Let* $m = 2^r \geq 2$. *There exists a map* $f : (S^{2k-1}, \lambda_{2m}) \to (S^{2k-1}, \lambda_m)$ *of degree* d *if and only if* $d = 2mj + \epsilon 2^k$, *where* $\epsilon = 1$ *for* $k \leq r$ *and* $\epsilon = 0$ *for* $k > r$.

Proof. The identity map is a map $(S^{2k-1}, \lambda_m) \to (S^{2k-1}, \lambda_m)$ of degree 1, and there is the map $s_k : (S^{2k-1}, \lambda_{2m}) \to (S^{2k-1}, \lambda_m)$, $m = 2^r \geq 2$, given by $s_k(z_1, \cdots, z_k) = 1/\sqrt{|z_1|^4 + \cdots + |z_k|^4}\,(z_1^2, \cdots, z_k^2)$, of degree 2^k. Both assertions (i), (ii) then follow from 1.1 (i)–(ii).

We say $\alpha \in \pi_{2k+1}S^{\ell}$ is *equivariant* if there exists a map $f : (S^{2k+1}, \lambda_4) \to (S^{\ell}, \lambda_2)$ with $[f] = \alpha$. For all $k \geq 1$, Lemma 1.2 (ii) determines all the equivariant elements of $\pi_{2k-1}S^{2k-1}$. The Z_4-action λ_4 on S^{2k-1} induces a Z_2-action $\bar{\lambda}_2$ on P^{2k-1}. Thus by naturality of the Hopf classification theorem all equivariant elements of $[P^{2k-1}, S^{2k-1}]$, i.e. those elements represented by maps $\bar{m}$ satisfying $\bar{m}\bar{\lambda}_2 = \lambda_2\bar{m}$, are also determined.

Conner and Floyd [4] use the Z_4-action on S^3 given by $\lambda_4'(z_1, z_2) = (-\bar{z}_2, \bar{z}_1)$. Since both λ_4, λ_4' are orthogonal actions, there exists an orthogonal map $(S^3, \lambda_4) \to (S^3, \lambda_4')$. Thus we may use either action.

Let $\beta \in \pi_3S^2 \cong Z$ be the generator represented by the Hopf construction of the complex multiplication $S^1 \times S^1 \to S^1$. Then $h_0(z_1, z_2) = z_1/z_2$ defines a map $(S^3, \lambda_4') \to (S^2, \lambda_2)$ representing the element $-\beta$. Here S^2 is the one-point compactification of the complex plane and $\lambda_2(z) = -\bar{z}^{-1}$ is the antipodal Z_2-action.

Theorem 1.3 $d\cdot\beta \in \pi_3S^2$ *is equivariant if and only if* $d = 4\ell - 1$ *for some* ℓ.

Proof. For any ℓ Lemma 1.2 (i) provides a map $f_\ell : (S^3, \lambda_4') \to (S^3, \lambda_4')$ of degree $4\ell + 1$. The composition $h_0 f_\ell$ then is a map $(S^3, \lambda_4') \to (S^2, \lambda_2)$ representing $(4\ell + 1)(-\beta) = (4(-\ell)-1)\beta \in \pi_3S^2$.

Conversely, let $\psi_q : \pi_nS^k \to \pi_nS^k$ be the homomorphism induced by left composition with a map $S^k \to S^k$ of degree q. Then ψ_q satisfies the property $H\psi_q = \psi_{q^2}H$, where $H : \pi_nS^k \to \pi_nS^{2k-1}$ denotes the Hopf homomorphism. $H : \pi_3S^2 \to \pi_3S^3$ is an isomorphism, so the relation $H(\lambda_2)_\# = H\psi_{-1} = \psi_1H = H$ implies that $(\lambda_2)_\# : \pi_3S^2 \to \pi_3S^2$ is the identity isomorphism. Because $\pi_2S^2 \simeq Z$, the other hypotheses of 1.1 (ii) obtain, so we may conclude that $[g] - [f] \in 4\pi_3S^2$ for any two maps $f, g : (S^3, \lambda_4') \to (S^2, \lambda_2)$.

$h_k(z_1, z_2) = (z_1/z_2)^{2k+1}$ defines a map $(S^3, \lambda_4') \to (S^2, \lambda_2)$ for every integer k. The subset $h_k^{-1}S^1 = \{(re(\theta_1), re(\theta_2)) \mid r = 1/\sqrt{2}\} \cong S^1 \times S^1$ is an equivariant torus in S^3 and the restriction $h_k \mid h_k^{-1}S^1$ defines a map $(S^1 \times S^1, \lambda_4') \to (S^1, \lambda_2)$ of type $(2k + 1, -(2k + 1))$. As h_k is the Hopf construction of its restriction $h_k \mid h_k^{-1}S^1$, h_k represents the element $-(2k + 1)^2\beta \in \pi_3S^2$. The maps h_k exhaust those equivariant elements of π_3S^2 represented by Hopf constructions of maps $(S^1 \times S^1, \lambda_4') \to (S^1, \lambda_2)$. More precisely,

Proposition 1.4 *There exists a map* $(S^1 \times S^1, \lambda_4') \to (S^1, \lambda_2)$ *of type* (m, n) *if and only if* $m = -n = 2k+1$ *for some* k.

Proof. For any k the map $h_k(e(\theta_1), e(\theta_2)) = e((2k + 1)(\theta_1 - \theta_2))$ is equivariant and has type $(2k + 1, -(2k + 1))$. Conversely, suppose $f : (S^1 \times S^1,$

$\lambda_4') \to (S^1, \lambda_2)$ has type (m, n). As $\lambda_2 \sim id$ we have $f\lambda_4' = \lambda_2 f \sim f$ and so $m = -n$. But Theorem 1.3 implies that no element of $2\pi_3 S^2$ is equivariant (a fact already proved in [4]), and the Hopf construction then shows that no type of the form $(2n, -2n)$ can have an equivariant representative. (Alternatively, the trivial fact that maps $\ell_j : (S^1, \lambda_4) \to (S^1 \times S^1, \lambda_4')$ exist representing $((2j+1)\iota, -(2j+1)\iota) \in \pi_1(S^1 \times S^1)$ together with Lemma 1.2 (ii) imply that types $(2n, -2n)$ do not have equivariant representatives.)

2. Codimensions 1 and 2

For $0 \leq i \leq k$ let

$$e_{2i} = \{(z_1, \cdots, z_{k+1}) \in S^{2k+1} \mid z_j = 0 \quad \text{for} \quad j > i+1, z_{i+1} = |z_{i+1}|\}$$

$$e_{2i+1} = \{(z_1, \cdots, z_{k+1}) \in S^{2k+1} \mid z_j = 0 \quad \text{for} \quad j > i+1, 0 \leq \arg z_{i+1} \leq \pi/2\}.$$

The collection $\{\lambda_4^j e_i \mid 0 \leq j \leq 3,\ 0 \leq i \leq k\}$ defines a cell decomposition of S^{2k+1} equivariant with respect to the cellular map λ_4. The subcomplex $S^{2j-1} \cup e_{2j} \cup \lambda_4 e_{2j}$ defines a sphere S^{2j}, and we have $S^\ell \subset S^{(\ell)} \subset S^{\ell+1}$, $\ell \leq 2k$, where $S^{(\ell)}$ denotes the ℓ-skeleton.

The Z_4-action λ_4 on $S^{(\ell)}$ induces a Z_2-action $\bar{\lambda}_2$ on the quotient space $P^{(\ell)} = S^{(\ell)}/\lambda_4^2$. $P^{(2k+1)}$ is the usual real projective space P^{2k+1} and $P^{(2k)} = P^{2k-1} \cup \bar{e}_{2k} \cup \bar{\lambda}_2 \bar{e}_{2k}$, where $\bar{e}_{2k}$ is the image of e_{2k} under the quotient map. Observe that $P^{(2k)} = P_1^{2k} \cup P_2^{2k}$, where $P_1^{2k} = P^{2k-1} \cup \bar{e}_{2k}$, $P_2^{2k} = P^{2k-1} \cup \bar{\lambda}_2 \bar{e}_{2k}$ are even dimensional real projective spaces and P_1^{2k} is the usual one in P^{2k+1}.

Now we may extend our definition of equivariant elements. We say $\alpha \in \pi_k S^\ell$ is *equivariant* if there exists a map fi: $S^k \subset (S^{(k)}, \lambda_4) \to (S^\ell, \lambda_2)$ with $[\text{fi}] = \alpha$.

LEMMA 2.1 (i) *$0 \in \pi_{n+k} S^n$ is equivariant if and only if some element of $\pi_{n+k+1} S^n$ is equivariant.*

(ii) *If $\pi_{n+k} S^n$ has an equivariant element, then so does $\pi_{m+k} S^m$ for all $m \geq n$.*

(iii) *If $0 \in \pi_{n+k-1} S^n$ is equivariant, then $0 \in \pi_{m+k-1} S^m$ is equivariant for all $m \geq n$.*

Proof. (i) Since $0 \in \pi_{n+k} S^n$ is equivariant, there exists an inessential map $mi : S^{n+k} \subset (S^{(n+k)}, \lambda_4) \to (S^n, \lambda_2)$. So we may extend mi over e_{n+k+1}, and then by equivariance over $S^{(n+k+1)}$ to a map m'. Now $[m'i]$ is an equivariant element of $\pi_{n+k+1} S^n$. Conversely, if $x \in \pi_{n+k+1} S^n$ is equivariant, then the restriction of any equivariant representative of x to S^{n+k} is equivariant and represents $0 \in \pi_{n+k} S^n$.

(ii) If $x \in \pi_{n+k} S^n$ is equivariant, then $i_{\#} x \in \pi_{n+k} S^{n+1}$, where $i : (S^{(n)}, \lambda_2) \subset (S^{(n+1)}, \lambda_2)$, is both equivariant and 0. This observation together with (i) and induction imply (ii).

(iii) is trivially implied by (i) and (ii).

Let $\pi_1 : S^{2k} \to P_1^{2k}$ be the usual quotient map, and let $p : S^{2k} \to S_1^{2k} \vee S_2^{2k}$ be the map collapsing S^{2k-1} to a point.

LEMMA 2.2. (i) *There exists a homotopy equivalence* $h : P^{(2k)} \to P_1^{2k} \vee S^{2k}$ *such that* $h\pi i : S^{2k} \subset S^{(2k)} \to P^{(2k)} \to P_1^{2k} \vee S^{2k}$ *is the map* $\pi_1 + (\pm) id$.

(ii) If $h_1 : P_1^{2k} \vee S^{2k} \to P^{(2k)}$ *is a homotopy inverse* of h, *then* $(h\bar{\lambda}_2 h_1)^*(g) = -g$ mod the summand $H^{2k}(P^{2k}; \boldsymbol{Z})$ where g is a generator of the summand $H^{2k}(S^{2k}; \boldsymbol{Z})$ in $H^{2k}(P_1^{2k} \vee S^{2k}; \boldsymbol{Z})$.

Proof. (i) Let $[P] \in P_1^{2k}$ be the basepoint, where $P = (0, \cdots, 0, 1) \in e_{2k} \subset S^{(2k)}$. We have $e_{2k} = S^{2k-1} * \{P\}$ where the join variable t is 0 at points $x \in S^{2k-1}$ and is 1 at the point P. Set

$$A = \{[x, t, P] \in e_{2k} \mid 0 \le t \le \tfrac{1}{2}\}$$
$$B = \{[x, t, P] \in e_{2k} \mid \tfrac{1}{2} \le t \le 1\}$$

so that $A \cup B = e_{2k}$. Furthermore, $p(A)$ and $p(B)$ are standard hemispheres of S_1^{2k} such that $p(A) \cap p(B) = S_1^{2k-1}$. In this notation the antipodal map of S_1^{2k} is given by $p[x, t, P] \to p[-x, 1 - t, P]$.

The attaching maps of the cells $\bar{e}_{2k}$, $\bar{\lambda}_2\bar{e}_{2k}$ are precisely the same map, so we may deform $P^{(2k)}$ by sliding the cell $\bar{\lambda}_2\bar{e}_{2k}$ off P_1^{2k} to form the homotopically equivalent space $P_1^{2k} \vee S^{2k}$. In fact, an explicit homotopy equivalence $h : P^{(2k)} \to P_1^{2k} \vee S^{2k}$ is given by

$$h(\pi[x, t, P]) = \pi[x, 2t - 1, P] \qquad \pi[x, t, P] \in \bar{e}_{2k}, \quad \tfrac{1}{2} \le t \le 1$$
$$h(\pi[x, t, P]) = \pi[x, -2t + 1, P] \quad \pi[x, t, P] \in \bar{e}_{2k}, \quad 0 \le t \le \tfrac{1}{2}$$
$$h \mid \bar{\lambda}_2\bar{e}_{2k} = \text{any relative homeomorphism of } (\bar{\lambda}_2\bar{e}_{2k}, \bar{\lambda}_2\dot{e}_{2k}) \text{ onto } (S^{2k}, [P]).$$

It is easy to check that the composition $h\pi i$ is the map $\pi_1 + (\pm)id$, using the above description of the antipodal map of S_1^{2k}.

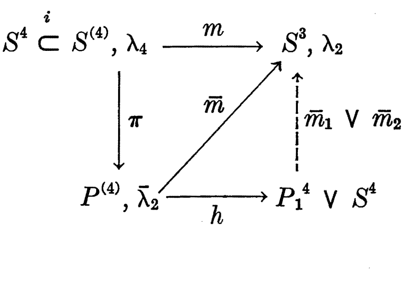
(ii) The assignment

$$[e_{2i+1} : \lambda_4^j e_{2i}] = -1, 1, 0, 0 \qquad \text{according as } j = 0, 1, 2, 3;$$
$$[e_{2i} : \lambda_4^j e_{2i-1}] = 1 \qquad \text{all } j$$

extends uniquely to a Z_4-invariant incidence function on S^{2k+1}. The cochains x_1, x_2 assuming values 1, 0, resp. 1, 1, on the cells $\bar{e}_{2k}$, $\bar{\lambda}_2\bar{e}_{2k}$ represent generators $\bar{x}_1$ (of infinite order), $\bar{x}_2$ (of order 2) of $H^{2k}(P^{(2k)}; \boldsymbol{Z}) \cong \boldsymbol{Z} + Z_2$. Assertion (ii) follows easily from the fact that $\bar{\lambda}_2 * x_1 = -\bar{x}_1 + \bar{x}_2$.

THEOREM 2.3. (Conner, Floyd [4]) $\ell \cdot \eta_3$ *is equivariant if and only if* ℓ *is odd.*

Proof. In the homotopy commutative diagram



which defines $\bar{m}_1 \vee \bar{m}_2$ up to homotopy, we have $[\bar{m}_1] = 0 \in [P_1^4, S^3]$, by the Steenrod classification theorem, and $[\bar{m}_2] = \eta_3$ by the essentiality of $\bar{m}$ [3, Theorem 3.12]. But then $[mi] = \eta_3$ from Lemma 2.2 (i).

The suspension $Sm : S^{2k+2} \to S^{\ell+1}$ of a map $m : (S^{2k+1}, \lambda_4) \to (S^\ell, \lambda_2)$ is equivariant with respect to λ_4 (defined on e_{2k+2}) and λ_2, and hence extends to a map $\tilde{m} : (S^{(2k+2)}, \lambda_4) \to (S^{\ell+1}, \lambda_2)$. As $\tilde{m}i = Sm$, $\tilde{m}$ is called the *equivariant suspension* of m.

The Z_4-actions $(S^{2(k+\ell)+3}, \lambda_4)$, $(S^{(2(k+\ell)+2)}, \lambda_4)$ can be viewed as the join Z_4-actions $(S^{2k+1}, \lambda_4)*(S^{2\ell+1}, \lambda_4)$, $(S^{2k+1}, \lambda_4)*(S^{(2\ell)}, \lambda_4)$, and $(S^{k+\ell+1}, \lambda_2) \cong (S^k, \lambda_2)*(S^\ell, \lambda_2)$. Thus for $f_i : (S^{(k_i)}, \lambda_4) \to (S^{\ell_i}, \lambda_2)$, $i = 1, 2$ and k_1, k_2 not both even, $f_1 * f_2$ defines a map $f_1*f_2 : (S^{(k_1+k_2+1)}, \lambda_4) \to (S^{\ell_1+\ell_2+1}, \lambda_2)$.

LEMMA 2.4. *There exists a map $\bar{m} : (P^{(2k)}, \bar{\lambda}_2) \to (S^{2k}, \lambda_2)$ representing the element of order 2 in $[P^{(2k)}, S^{2k}] \cong \boldsymbol{Z}_2 + \boldsymbol{Z}$ if and only if $k = 1$.*

Proof. Let $\bar{m}_1 = \bar{m} \mid P^{(2)}$ for any map $\bar{m} : (P^3, \bar{\lambda}_2) \to (S^2, \lambda_2)$. $[\bar{m}_1]$ is in the kernel of $(\pi i)^* : [P^{(2)}, S^2] \to [S^2, S^2]$, since $\bar{m}_1\pi i$ extends to $\bar{m}\pi$ over S^3. As coindex $P^{(2)} = 2$, $\bar{m}_1$ must be essential ([3] Theorem 3.12) and so $[\bar{m}_1]$ is the element of order 2.

By cellular approximation the induced map $\bar{\bar{m}} : P_4^{2k} = P^{(2k)}/\bar{\lambda}_2 \to P^{2k}$ of any given map $\bar{m} : (P^{(2k)}, \bar{\lambda}_2) \to (S^{2k}, \lambda_2)$ is homotopic to a cellular map. By the covering homotopy property $\bar{m}$ is homotopic to a cellular map $\bar{m}_1$. The determination of the equivariant elements of $\pi_{2k-1}S^{2k-1}$ (Lemma 1.2 (i)) and the naturality of the Hopf classification theorem imply that $[\bar{m}_1'] = [\bar{m}_1 \mid P^{2k-1}]$ must correspond to N times a generator of $H^{2k-1}(P^{2k-1}; \boldsymbol{Z})$, where N is odd or even according as $k = 1$ or $k > 1$. In the commutative diagram

$$\begin{array}{ccccc}
H^{2k-1}S^{2k-1} & \xrightarrow{\delta_1} & H^{2k}(S^{2k}, S^{2k-1}) & \xrightarrow{j_1^*} & H^{2k}S^{2k} \\
\Big\downarrow {\bar{m}'_1}^* & & \Big\downarrow f & & \Big\downarrow \bar{m}_1^* \\
H^{2k-1}P^{2k-1} & \xrightarrow{\delta_2} & H^{2k}(P^{(2k)}, P^{2k-1}) & \xrightarrow{j_2^*} & H^{2k}P^{(2k)}
\end{array}$$

we may select generators $g; g_1, \lambda_2^* g_1; \hat{g}$ for the top line such that $\delta_1(g) = g_1 + \lambda_2^* g_1$, $j_1^* g_1 = -j_1^* \lambda_2^* g_1 = \hat{g}$; and generators $h; h_1, \bar{\lambda}_2^* h_1; \bar{x}_1, \bar{x}_2$ such that $\delta_2 h = 2(h_1 + \bar{\lambda}_2^* h_1)$, $j_2^* h_1 = \bar{x}_1$, $j_2^*(h_1 + \bar{\lambda}_2^* h_1) = \bar{x}_2$ (the element of order 2). Now $\delta_2(\bar{m}_1')^*(\dot{g}) = 2N(h_1 + \bar{\lambda}_2^* h_1)$, so if $f(g_1) = ah_1 + b\bar{\lambda}_2^* h_1$, then $f(g_1 + \lambda_2^* g_1) = (a + b)(h_1 + \bar{\lambda}_2^* h_1) = 2N(h_1 + \bar{\lambda}_2^* h_1)$. N is even when $k > 1$, so $a = b =$ an odd integer cannot occur. Thus the element of order 2 in $H^{2k}(P^{(2k)}; \boldsymbol{Z})$ is not in the image of $\bar{m}_1^*$.

THEOREM 2.5. *Let $k \geq 1$*

(i) *$\ell \cdot \eta_{2k+2}$ is equivariant if and only if ℓ is even.*

(ii) *$\ell \cdot \eta_{4k+1}$ is equivariant if and only if ℓ is even.*

Proof. (i) For $k \geq 1$ Lemma 2.4 implies that any map $m : (S^{2k+3}, \lambda_4) \to (S^{2k+2}, \lambda_2)$ induces a map $\bar{m}$ whose restriction to P_1^{2k+2} is inessential. Hence m

is homotopic to some map $\tilde{m}q\pi : S^{2k+3} \to P^{2k+3} \to P^{2k+3}/P_1^{2k+2} = S^{2k+3} \to S^{2k+2}$ and as deg $q\pi = \pm 2$, (i) follows.

(ii) Any representative $f : S^\ell \to S^{\ell-1}$ of $\eta_{\ell-1}$ induces an isomorphism $f^* : \widetilde{KO}^{\ell-2}S^{\ell-1} \to \widetilde{KO}^{\ell-2}S^\ell$. In fact, since $S^{\ell-1} \cup_f e^{\ell+1} \sim \Sigma^{\ell-3}CP^2$, we have an exact sequence

$$\widetilde{KO}^{\ell-2}S^{\ell-1} \xrightarrow{\delta} \widetilde{KO}^{\ell-1}S^{\ell+1} \to \widetilde{KO}^{\ell-1}(\Sigma^{\ell-3}CP^2)$$

where $\delta = s^*f^*$ and s^* is the suspension isomorphism. As $\widetilde{KO}^{\ell-1}(\Sigma^{\ell-3}CP^2) \cong \widetilde{KO}^2(CP^2)$ is free abelian [7], f^* must be an isomorphism.

Apply $\widetilde{KO}^{4k}$ to the diagram

$$\begin{array}{ccc} S^{4k+2} \overset{i}{\subset} S^{(4k+2)}, \lambda_4 & \xrightarrow{\quad m \quad} & S^{4k+1}, \lambda_2 \\ & \searrow^{\pi} \quad \nearrow_{\bar{m}} & \\ P_a^{4k+2} + P_b^{4k+2} & \xrightarrow{\quad j \quad} & P^{(4k+2)}, \bar{\lambda}_2 \end{array}$$

(+ is disjoint union)

to obtain the commutative diagram

$$\begin{array}{ccc} \widetilde{KO}^{4k}S^{4k+1} & \xrightarrow{\quad m^* \quad} & \widetilde{KO}^{4k}S^{4k+2} \\ & \searrow^{\bar{m}^*} \quad \nearrow_{(\pi\, i)^*} & \\ \widetilde{KO}^{4k}P^{(4k+2)} & \xrightarrow{\quad j^* \quad} & \widetilde{KO}^{4k}P_a^{4k+2} \oplus \widetilde{KO}^{4k}P_b^{4k+2}. \end{array}$$

The map j sends P_a^{4k+2}, P_b^{4k+2} homeomorphically onto P_1^{4k+2}, P_2^{4k+2} respectively. If $j_i : P^{4k+1} \to P_i^{4k+2}$ is the usual inclusion, then $j_i^* : \widetilde{KO}^{4k-1}P_i^{4k+2} \to \widetilde{KO}^{4k-1}P^{4k+1}$ is epic [7], and so from the Mayer-Vietoris sequence, j^* is monic.

The requirement $\bar{\lambda}_2 j = jT$ defines a map $T : P_a^{4k+2} + P_b^{4k+2} \to P_a^{4k+2} + P_b^{4k+2}$. $\widetilde{KO}^{4k}P^{4k+2}$ is cyclic of order $4N$ [7] and so from Lemma 2.2 $\widetilde{KO}^{4k}P^{(4k+2)} \cong Z_{4N} + Z_2$. We may select generators h_a, h_b of $\widetilde{KO}^{4k}P_a^{4k+2}$, $\widetilde{KO}^{4k}P_b^{4k+2}$, and generators g_1, g_2 of $\widetilde{KO}^{4k}P^{(4k+2)}$ such that $T^*h_a = h_b$, $j^*g_1 = h_a \oplus h_b$ and $j^*g_2 = 2Nh_a \oplus 0$. Hence $\bar{\lambda}_2^* g_1 = g_1$ while $\bar{\lambda}_2^* g_2 = 2Ng_1 + g_2$. So if $\bar{m}^*g = n_1g_1 + n_2g_2$ for $g \neq 0$ in $\widetilde{KO}^{4k}S^{4k+1}$, then $\lambda_2^* =$ identity and $\bar{m}^*\bar{\lambda}_2^* = \lambda_2^*\bar{m}^*$ imply $n_1g_1 + n_2g_2 = (n_1 + 2Nn_2)g_1 + n_2g_2$ in $\widetilde{KO}^{4k}P^{(4k+2)}$. Hence $n_1 \equiv n_1 + 2Nn_2 \pmod{4N}$ and so n_2 is even. However n_1 must also be even, since the map $\bar{m} \mid P^{4k+1}$ is inessential and the kernel of $i^* : \widetilde{KO}^{4k}P^{(4k+2)} \to \widetilde{KO}^{4k}P^{4k+1}$ is contained in $2KO^{4k}P^{(4k+2)}$. Hence $i^*m^* = 0$ and consequently, mi is inessential.

$0 \in \pi_5 S^4$ is equivariant and so by Lemma 2.1 (iii) $0 \in \pi_{n+1}S^n$ is equivariant for all $n \geq 4$. By Lemma 2.1 (i) $\pi_{n+2}S^n$ has an equivariant element for all $n \geq 4$.

THEOREM 2.6. *Let $k \geq 1$.*

(i) *$\ell \cdot \eta_{4k}^2 = \ell\eta_{4k+1}\eta_{4k}$ is equivariant if and only if ℓ is even.*

(ii) *Both elements* $0\cdot\eta_{4k+1}^2$ *and* η_{4k+1}^2 *are equivariant. Similarly both elements* $0\cdot\eta_{4k+2}^2$ *and* η_{4k+2}^2 *are equivariant.*

(iii) $\ell\cdot\eta_{4k+3}^2$ *is equivariant if and only if* ℓ *is even.*

Proof. We use the notations T, j, h_a, h_b, g_1, g_2 defined in the proof of Theorem 2.5 (ii). Recall that $\widetilde{KO}^{4k}P^{4k+2}$ is cyclic of order $4N$. As λ_2 is defined on an even-dimensional sphere, $\lambda_2{}^*g = -g$ for a generator g of $\widetilde{KO}^{4k}S^{4k}$. If $\bar{m}^*g = n_1g_1 + n_2g_2$ where $\bar{m} : (P^{(4k+2)}, \bar{\lambda}_2) \to (S^{4k}, \lambda_2)$, then $\bar{\lambda}_2{}^*\bar{m}^* = \bar{m}^*\lambda_2{}^* = -\bar{m}^*$ implies that

$$n_1g_1 + n_2g_2 + (n_1g_1 + 2Nn_2g_1 + n_2g_2) = (2n_1 + 2Nn_2)g_1 + 2n_2g_2 = 0$$

in $\widetilde{KO}^{4k}P^{(4k+2)}$. Hence both n_1 and n_2 are even and image $\bar{m}^* \subset 2\cdot\widetilde{KO}^{4k}P^{(4k+2)}$ Hence $i^*\pi^*\bar{m}^* = 0$ and by Adams [1], $\bar{m}\pi i = mi$ is inessential.

(ii) First $0\cdot\eta_4^2$ is equivariant and so $0\cdot\eta_n^2$ is equivariant for all $n \geq 4$. To show that η_{4k+1}^2 is equivariant, consider any inessential map $\bar{m} : (P^{4k+1}, \bar{\lambda}_2) \to (S^{4k+1}, \lambda_2)$ (such exist by Lemma 1.2 (ii)). Using the homotopy extension property, we extend $\bar{m}$ over P_1^{4k+2} to a representative $\hat{m}_1$ of the generator of $\pi^{4k+1}P_1^{4k+2} \simeq Z_2$ [11]. We further extend $\hat{m}_1$ by equivariance to a map $\bar{m}_1 : (P^{(4k+2)}, \bar{\lambda}_2) \to (S^{4k+1}, \lambda_2)$. By Theorem 2.5 (ii) $\bar{m}_1\pi i$ is inessential, so $m = \bar{m}_1\pi$ extends to a map $m_2 : (S^{4k+3}, \lambda_4) \to (S^{4k+1}, \lambda_2)$.

We assert that $[m_2] = \eta_{4k+2}^2$. There is the homotopy commutative diagram

$$\begin{array}{ccc} S^{4k+3}, \lambda_4 & \xrightarrow{\quad m_2 \quad} & S^{4k+1}, \lambda_2 \\ \Big\downarrow \pi & \overset{\bar{m}_2}{\nearrow} & \Big\uparrow \bar{m}_a \vee \bar{m}_b \\ P^{4k+3}, \bar{\lambda}_2 & \xrightarrow[p]{\quad} & P^{4k+3}/P^{4k+1} \sim S^{4k+2} \vee S^{4k+3} \end{array}$$

since $\bar{m}_2 \mid P^{4k+1}$ is inessential. As $\hat{m}_1$ represents a generator of $\pi^{4k+1}P_1^{4k+2}$, $\hat{m} \sim f\circ p_1 : P_1^{4k+2} \to P_1^{4k+2}/P^{4k+1} \simeq S^{4k+2} \to S^{4k+1}$ with $[f] = \eta_{4k+1}$ [11]. Hence $[\bar{m}_a] = \eta_{4k+1}$. Also $Sq^2 \neq 0$ on $H^{4k+2}(P^{4k+4}; \boldsymbol{Z}_2)$ and $H^{4k+2}(P^{4k+4}; \boldsymbol{Z}) \cong Z_2$ imply that $[p\pi] = \eta_{4k+2} + (\pm 2)\iota$. Hence $[m_2] = \eta_{4k+1}\eta_{4k+2} + 2[\bar{m}_b] = \eta_{4k+1}^2$.

By equivariant suspension the result for $\pi_{4k+4}S^{4k+2}$ follows immediately.

(iii) For any $\bar{m} : (P^{4k+5}, \bar{\lambda}_2) \to (S^{4k+3}, \lambda_2)$, $\bar{m} \mid P^{4k+3}$ is inessential. This provides a homotopy commutative diagram

$$\begin{array}{ccc} S^{4k+5}, \lambda_4 & \xrightarrow{\quad m \quad} & S^{4k+3}, \lambda_2 \\ \Big\downarrow \pi & \overset{\bar{m}}{\nearrow} & \Big\uparrow \\ P^{4k+5}, \bar{\lambda}_2 & \xrightarrow{\quad p \quad} & P^{4k+5}/P^{4k+3} \sim S^{4k+4} \vee S^{4k+5}. \end{array}$$

In this case $Sq^2 = 0$ on $H^{4k+4}(P^{4k+6}; \boldsymbol{Z}_2)$ and so $[p\pi] = 0\cdot\eta_{4k+4} + (\pm 2)\iota$. Hence $[m] = 0$.

3. Codimension 3

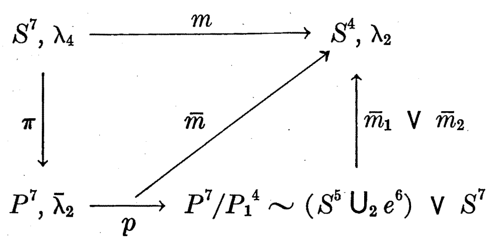
We write $(s, t) \equiv (m, n) \pmod{(p, q)}$ if $s \equiv m \pmod p$ and $t \equiv n \pmod q$.

THEOREM 3.1. (i) *$s\nu_4 + t\omega \in \pi_7 S^4$ is equivariant if and only if $(s, t) \equiv (2, 5)$, $(6, 3)$ or $(10, 1) \pmod{(12, 6)}$.*

(ii) *Let $j \geq 5$. If $\ell \cdot \nu_j$ is equivariant, then $\ell \equiv 0 \pmod{12}$.*

(iii) *Let $j \geq 5$. If $j \neq 6, 7$, then both $0 \cdot \nu_j$ and $12 \cdot \nu_j$ are equivariant, and if $j = 6$ or 7, then $0 \cdot \nu_j$ is equivariant.*

Proof. (i) For any $m : (S^7, \lambda_4) \to (S^4, \lambda_2)$ Lemma 2.3 implies that the restriction $\bar m \mid P^{(4)}$ of the induced map $\bar m$ is inessential, and so we have a homotopy commutative diagram



where the homotopy equivalence $\sim$ is given in [9]. Also in [9] is the result that $[p\pi] = \tilde\eta + (\pm 2)\iota$. As coindex $P^5 = 4$, $\bar m \mid P^5$ must be essential ([13] Theorem 3.12), and so $[\bar m_1] = \pm\tilde\eta$ [2]. Hence if $[\bar m_2] = d_1\nu_4 + d_2\omega$, then $[m] = \pm\tilde\eta \circ \tilde\eta + 2\,(d_1\nu_4 + d_2\omega) = 2\,d_1\nu_4 + (2\,d_2 \pm 3)\omega$. (Here $\tilde\eta$ is chosen so that $\tilde\eta \circ \tilde\eta = 3\omega$.)

Next $\lambda_2 m = m\lambda_4 \sim m$, since $\lambda_4 \sim id$. Also, as is well known, the homomorphism $\lambda_{2\#} : \pi_7 S^4 \to \pi_7 S^4$ is given by $\lambda_{2\#}(\nu_4) = \nu_4 - \omega$, $\lambda_{2\#}(\omega) = -\omega$ and so we have

$$[m] = 2\,d_1\nu_4 + (2\,d_2 \pm 3)\,\omega = [\lambda_2 m] = \lambda_{2\#}[m] = 2\,d_1\nu_4 + (-2\,d_1 - 2\,d_2 \mp 3)\,\omega.$$

Hence the relation $2\,d_1 + 4\,d_2 \equiv 6 \pmod{12}$. But then d_1 must be odd, from which we easily deduce that $(2\,d_1, 2\,d_2 + 3) \equiv (2, 5)$, $(6, 3)$ or $(10, 1)$ (mod $(12, 6)$).

Conversely, there exists a map $g : (S^7, \lambda_4) \to (S^4, \lambda_2)$, say $[g] = s\nu_4 + t\omega$ with s, t satisfying the stated condition. Note for $\alpha = \nu_4$, $\sum_{j=0}^{3} (\lambda_2^{\,j})_\#\alpha = 4\nu - 2\omega$ and so for any other pair (s_1, t_1) satisfying the condition, there exists an integer N such that $s_1\nu_4 + t_1\omega = [g] + \sum_{j=0}^{3} (\lambda_2^{\,j})_\#(N\nu_4)$. Hence Folkman's result Proposition 1.1 (i) implies that $s_1\nu_4 + t_1\omega$ is equivariant also.

(ii) The homomorphism $(\lambda_2)_\# : \pi_{2k+1}(S^{2k-2}) \to \pi_{2k+1}(S^{2k-2})$, $k \geq 4$, is multiplication by -1. Hence $[m] = [m\lambda_4] = [\lambda_2 m] = -[m]$ and so $2[m] = 0$. In $\pi_{2k+1}S^{2k-2} \cong Z_{24}$, this means that $[m] = \ell \cdot \nu_{2k-2}$ where $\ell \equiv 0 \pmod{12}$.

The self map $\bar\lambda_2{}' = h\bar\lambda_2 h_1$ of $P_1^{2k+2} \vee S^{2k+2}$, where $h : P^{(2k+2)} \to P_1^{2k+2} \vee S^{2k+2}$ is a homotopy equivalence and h is a homotopy inverse of h_1, induces the nontrivial isomorphism on the summand $H^{2k+2}(S^{2k+2}; \boldsymbol{Z})$ mod $H^{2k+2}(P^{2k+2}; \boldsymbol{Z})$ by

Lemma 2.2 (ii). In the homotopy commutative diagram

$$\begin{array}{ccc} S^{2k+2} \overset{i}{\subset} S^{(2k+2)}, \lambda_4 & \xrightarrow{m} & S^{2k-1}, \lambda_2 \\ \pi_1 + \pi_2 \searrow & \downarrow h\pi & \nearrow \bar{m}h_1 = \bar{m}_1 \vee \bar{m}_2 \\ & P_1^{2k+2} \vee S^{2k+2} & \end{array}$$

the equivariance condition $\bar{m}\bar{\lambda}_2 = \lambda_2\bar{m}$ implies $\bar{m}h_1\bar{\lambda}_2' \sim \lambda_2\bar{m}h_1$ and so $2[\bar{m}_2] = 0$. Moreover, from Rees [9] we have $2[\bar{m}\pi_1] = 0$. Hence $2[mi] = 2([\bar{m}_1\pi_1] + [\bar{m}_2\pi_2]) = 0$.

(iii) The equivariant suspensions of the equivariant elements $2\nu_4 + 5\omega$, $14\nu_4 + 5\omega$ are the elements $12\nu_5$, $0 \cdot \nu_5$. The join of $2\nu_4 + 5\omega$ and $2\iota_1$ is $0 \cdot \nu_6$, while for $j \geq 5$ the join of $0 \cdot \nu_j$ and $2\iota_1$ is $0 \cdot \nu_{j+2}$. Finally $12\nu_{4k}$ (resp. $12\nu_{4k+2}$) is the join of η_{4k-3}^2 (resp. η_{4k-1}^2) and $(2\ell + 1)\eta_2$ (resp. $(2\ell + 1)\eta_2$).

4. Further results

Instead of seeking the least k for given n for which there exists a map $(P^{(n)}, \bar{\lambda}_2) \to (S^k, \lambda_2)$, we can fix k and ask for the largest n that such a map exists. The latter point of view is suggested by the obstruction theory for extending equivariant maps [6, §2]. For $k \leq 3$ this n has been determined in [4]. The following result extends this information somewhat.

THEOREM 4.1. (i) *$\ell \cdot \omega_5 = \ell \cdot [\iota_5, \iota_5] \in \pi_9 S^5$ is equivariant if and only if ℓ is even.*

(ii) *$\ell \cdot \nu\eta^2 \in \pi_{10} S^5$ is equivariant if and only if ℓ is odd.*

(iii) *Infinitely many elements of $\pi_{11} S^6$ are equivariant, and no element of $\{N \cdot \omega_6 \in \pi_{11} S^6 \mid N \text{ odd}\}$ is equivariant.*

(iv) *$0 \in \pi_{13} S^7$ is equivariant; $0 \in \pi_{14} S^7$ is not equivariant.*

(v) *Infinitely many elements of $\pi_{15} S^8$ are equivariant.*

Proof. (i) Randall [8] has shown that ω_5 is not even projective, i.e. ω_5 admits no representative of the form $f\pi : S^9 \to P^9 \to S^5$. Since $0 \in \pi_8 S^5$ is equivariant, the only other element of $\pi_9 S^5$, namely 0, must be equivariant.

(ii) (i) implies that some element of $\pi_{10} S^5$ is equivariant. If $0 \in \pi_{10} S^5$ is equivariant, then some element x of $\pi_{11} S^5$ is also equivariant. But then the join construction implies that the join $x * x$ of x with itself in $\pi_{23} S^{11}$ is equivariant. However $x * x = 0$ for all elements x in $\pi_{11} S^5$ [10] and so some element of $\pi_{k+13} S^k$ is equivariant for all $k \geq 11$. For $k = 12$ this contradicts results of [5], and so (ii) is proved.

(iii) By (ii) we have that $\pi_{k+5} S^k$ has an equivariant element for all $k \geq 5$. If $x \in \pi_{11} S^6$ is equivariant, then so is $x + 4y$ for any $y \in \pi_{11} S^6$, because $(\lambda_2)_\# : \pi_{11} S^6 \to \pi_{11} S^6$ is the identity isomorphism and we may apply Folkman's Prop. 1.1 (i). This Proposition also establishes the second assertion of (iii), since ω_6 (and also $-\omega_6$ by an identical argument) is not projective [8].

(iv) Using (iii) we have an inessential map $im : (S^{11}, \lambda_4) \to (S^6, \lambda_2) \to (S^7, \lambda_2)$, and hence a map $m_1 i : S^{12} \subset (S^{(12)}, \lambda_4) \to (S^7, \lambda_2)$. $[m_1 i] = 0$ since $\pi_{12} S^7 = 0$, so m_1 extends to a map $m_2 : (S^{13}, \lambda_4) \to (S^7, \lambda_2)$ which by construction satisfies the homotopy commutative diagram

$$\begin{array}{ccc} S^{13} & \xrightarrow{\ m_2\ } & S^7 \\ \pi \downarrow & & \uparrow \\ P^{13} & \xrightarrow[p]{} & P^{13}/P^{11} \sim S^{12} \vee S^{13} \end{array}$$

Now $H^{14}(P^{14}; \boldsymbol{Z}) = \boldsymbol{Z}_2$ and $Sq^2 = 0$ on $H^{12}(P^{14}; \boldsymbol{Z}_2)$ so $[p\pi] = 0 \cdot \eta_{12} \pm 2\iota$. But then $[m_2] = 0$ since $\pi_{13} S^7 = \boldsymbol{Z}_2$. If $0 \in \pi_{14} S^7$ were equivariant, some element y of $\pi_{15} S^7$ would be also. But then the join $y * y = 0 \in \pi_{31} S^{15}$ [10] would be equivariant, thus implying that $\pi_{k+17} S^k$ contains an equivariant element for all $k \geq 15$, a contradiction for $k = 16$ [15].

(v) By 1.1 (*i*) it suffices to show that some element of $\pi_{15} S^8$ is equivariant. As some element of $\pi_{14} S^7$ is equivariant, this is implied by Lemma 2.1 (i).

Our results on equivariant maps give the following table for the coindex of $P^{(n)}$:

n	2, 3	4	5, 6, 7	8, 9, 10	11	12	13	14	15	16
coindex $P^{(n)}$	2	3	4	5	6	6 or 7	7	7	8	8 or 9

For $n \leq 6$ these results were first given in [4].

Some open questions in low codimension: Is η_{4k+3} equivariant ($k \geq 1$)? Are $12\nu_6$, $12\nu_7$ equivariant? Is $0 \in \pi_{11} S^6$ equivariant? The answers to these questions would determine all equivariant elements in $\pi_{n+k} S^n$ of codimension ≤ 5 for all n.

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