LIMITS OF INTEGRALS INVOLVING n-TUPLY PERIODIC FUNCTIONS

By José A. Canavati

1. Introduction

The object of this paper is to give a generalization of the following result due to L. Fejér [4]. (See also [2], p. 67).

Fejérs Lemma: *If* $f \epsilon L^1 (]-\pi, \pi[, dx)$ *has period* 2π *and* g *is a bounded measurable function of period* 2π, *then*

$$(1.1) \qquad \lim_{t\to\pm\infty} \int_{-\pi}^{\pi} g(tx)f(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\, dx \int_{-\pi}^{\pi} f(x)\, dx.$$

The result we prove here (Theorem 2.1) is stated under the more general hypothesis that g and f are functions defined on R^n, with g an n-tuply periodic essentially bounded function and f an integrable function. There is no periodicity condition on f. The idea of the proof of Theorem 2.1 is essentially the one given in [2]: One checks formula (2.2) for a dense class of functions in $L^1 (R^n, dx)$, and then one proves the general case by taking limits. But unlike in [2], where the class of all step functions in $]-\pi, \pi[$ is used, we found more convenient for the proof of our general result, to use the class of all continuous functions in R^n with compact support. However, we would like to remark that we could have used the class of all step functions on R^n as well. Another consequence of Theorem 2.1 will be the well known:

Riemann-Lebesgue lemma: If $f \epsilon L^1 (R^n, dx)$ *then its Fourier transform*

$$(1.2) \qquad \hat{f}(t) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} f(x) \exp(-it.x)\, dx, \qquad t.x = t_1 x_1 + \cdots + t_n x_n$$

is a function which vanishes at infinite i.e.,

$$(1.3) \qquad \lim_{|t|\to\infty} \hat{f}(t) = 0,$$

where $|t|$ *denotes the Euclidean norm of the vector* $t = (t_1, \cdots, t_n)$.

In section 3 we rephrase Theorem 2.1 within the framework of probability theory (Corollary 3.1) and explore some of its consequences.

2. The Main Result

Let us recall that for a real or complex valued function Φ on R^n a "period" is a vector $p \epsilon R^n$ such that $\Phi(x + p) = \Phi(x)$, $x \epsilon R^n$.

It is well known [3] that the set of periods of Φ forms an additive subgroup of R^n, which is closed if Φ is continuous, and there are $(n + 1)(n + 2)/2$ categories of such closed subgroups of R^n. If $p_1, \cdots, p_n$ are n linearly independent vectors in R^n, the set of all linear combinations $m_1p_1 + \cdots + m_np_n$ with integer coefficients $m_1, \cdots, m_n$ represents an important subgroup. A function on R^n where each element of such subgroup is a period is called "n-tuply periodic" i.e., Φ satisfies.

$$\Phi(x + m_1p_1 + \cdots + m_np_n) = \Phi(x),\ x \epsilon R^n \tag{2.1}$$

where the m_i's are arbitrary integers.

THEOREM 2.1: *Suppose Φ is an n-tuply periodic function on R^n and let*

$$B = \{r_1p_1 + \cdots + r_np_n \mid 0 \leq r_i \leq 1,\ i = 1, 2, \cdots, n\}$$

be the "fundamental parallelepiped" spanned by the periods $p_1, \cdots, p_n$. If $\Phi \epsilon L^\infty(R^n, dx)$, then for every $f \epsilon L^1(R^n, dx)$ one has

$$\lim_{m(t)\to\infty} \int_{R^n} \Phi_t(x) f(x)\, dx = \frac{1}{|B|} \int_B \Phi(x)\, dx \int_{R^n} f(x)\, dx, \tag{2.2}$$

where $\Phi_t(x) = \Phi(t_1x_1, \cdots, t_nx_n)$, $t = (t_1, \cdots, t_n)$, $m(t) = \min t_i$, $1 \leq i \leq n$ and $|B|$ denotes the n-dimensional volume of the fundamental parallelepiped B.

Proof: Without loss of generality we can assume that Φ and f are real valued functions, also that

$$\int_B \Phi(x)\, dx = 0, \tag{2.3}$$

considering the n-tuply periodic function $\Psi(x) = \Phi(x) - c$, where

$$c = \frac{1}{|B|} \int_B \Phi(x)\, dx.$$

The proof is divided in two steps.

Step I: Suppose first that

$$p_1 = e_1 = (1, 0, \cdots, 0), \cdots, p_n = e_n = (0, \cdots, 0, 1). \tag{2.4}$$

If $f = \phi$, where ϕ is a continuous function with compact support, then there is an integer $N > 0$ such that

$$\text{supp}\,(\phi) \subseteq\ ]-N, N[\ \times \cdots \times\ ]-N, N[\ = Q_N. \tag{2.5}$$

For $i = 1, 2, \cdots, n$ we partition the interval $[-N, N]$ into the subintervals

$$\begin{aligned} &[-N, -N[t_i]/t_i],\ [k_i/t_i, (k_i + 1)/t_i],\ k_i = -N[t_i], \cdots, \\ &\qquad\qquad - 1, 0, 1, \cdots, N[t_i] - 1,\ [N[t_i]/t_i, N], \end{aligned} \tag{2.6}$$

where $[t]$ denotes the "greatest integer" function.

Note that the first and last intervals in (2.6) are reduced to a point when t_i is a positive integer and, in this case, (2.6) gives a partition of $[-N, N]$ into $2Nt_i$ subintervals of equal length. Also since

$$\lim_{t\to\infty} [t]/t = 1$$

we have

$$\text{supp}\,(\phi) \subseteq \left]-N[t_1]/t_1\,,\ N[t_1]/t_1\right[\times \cdots \times \left]-N[t_n]/t_n\,, N[t_n]/t_n\right[\tag{2.7}$$

when $m(t) > 0$ is large enough.

Now (2.6) induces a subdivision of the closure $\bar{Q}_N$ of Q_N into closed subblocks $Q_{k_1,\dots,k_n}$ whose interiors are pairwise disjoint

$$\bar{Q}_N = \cup_{k_1,\dots,k_n} Q_{k_1,\dots,k_n}\,. \tag{2.8}$$

If all the t_i's are positive integers, then all the subblocks $Q_{k_1,\dots,k_n}$ in (2.8) have volume equal to

$$|Q_{k_1,\dots,k_n}| = \frac{1}{t_1 t_2 \cdots t_n}\,.$$

Since $\Phi \epsilon L^\infty(R^n, dx)$, there is a number $a \epsilon R$ such that

$$\Phi(x) \geq a \text{ a.e. in } R^n. \tag{2.9}$$

Hence if we let

$$I(t) = \int_{R_n} \Phi_t(x)\phi(x)\, dx, \tag{2.10}$$

then we have

$$\begin{aligned} I(t) &= \int_{\bar{Q}_N} \Phi_t(x)\phi(x)\, dx = \sum\nolimits_{k_1,\dots,k_n} \int_{Q_{k_1,\dots,k_n}} \Phi_t(x)\phi(x)\, dx \\ &= \sum\nolimits_{k_1,\dots,k_n} \int_{Q_{k_1,\dots,k_n}} [\Phi_t(x) - a]\phi(x)\, dx + \sum\nolimits_{k_1,\dots,k_n} \int_{Q_{k_1,\dots,k_n}} a\phi(x)\, dx. \end{aligned}$$

From (2.9) and the mean value theorem for integrals [5] we have

$$I(t) = \sum\nolimits_{k_1,\dots,k_n} \phi(y_{k_1,\dots,k_n}) \int_{Q_{k_1,\dots,k_n}} [\Phi_t(x) - a]\, dx + a \int_{\bar{Q}_N} \phi(x)\, dx,$$

where $y_{k_1,\dots,k_n} \epsilon Q_{k_1,\dots,k_n}$.

If we make use of (2.1), (2.3), (2.4) and (2.7) we see that for $m(t) > 0$ large enough one has

$$I(t) = -\, a \sum\nolimits_{k_1,\dots,k_n} \phi(y_{k_1,\dots,k_n})\, |Q_{k_1,\dots,k_n}| + a \int_{\bar{Q}_N} \phi(x) dx,$$

and since ϕ is Riemann integrable on $\bar{Q}_N$ we must have

$$\lim_{m(t)\to\infty} I(t) = 0.$$

Suppose now that $f \epsilon L^1(R^n, dx)$ is any integrable function. Since the set of all continuous functions ϕ with compact support is dense in $L^1(R^n, dx)$ [6]. Given $\epsilon > 0$ there is a such ϕ with

$$\|f - \phi\|_1 = \int_{R^n} |f(x) - \phi(x)|\, dx < \frac{\epsilon}{2\|\Phi\|_\infty}\,. \tag{2.11}$$

Using (2.11) we obtain

$$\begin{aligned} \left| \int_{R^n} \Phi_t(x) f(x)\, dx \right| &\leq \left| \int_{R^n} \Phi_t(x)[f(x) - \phi(x)]\, dx \right| + \left| \int_{R^n} \Phi_t(x)\phi(x)\, dx \right| \\ &\leq \|\Phi\|_\infty \|f - \phi\|_1 + \left| \int_{R^n} \Phi_t(x)\phi(x)\, dx \right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \text{if} \quad m(t) > 0 \text{ is large enough.} \end{aligned}$$

Step II: Suppose now that the linearly independent periods $p_1, \cdots, p_n$ are arbitrary. Then there is a non-singular linear transformation $T: R^n \to R^n$ such that $T(e_i) = p_i, 1 \leq i \leq n$. From the change of variables formula for Lebesgue integrals [5] we have

$$\int_{R^n} \Phi_t(x) f(x)\, dx = |\, det\, T\,| \int_{R^n} \Phi_t \circ T(x) f \circ T(x)\, dx.$$

Clearly $\Phi_t \circ T$ is an n-tuply periodic function satisfying the hypothesis of Step I and $f \circ T \in L^1(R^n, dx)$. Thus (2.2) holds.

As a consequence of the previous theorem we have the following result which includes Fejérs lemma and the one-dimensional Riemann-Lebesgue lemma as particular cases:

COROLLARY 2.2: *Suppose Φ is a periodic function on R with period $p > 0$. If $\Phi \in L^\infty(R, dx)$, then for every $f \in L^1(R, dx)$ one has*

$$\lim_{t \to \pm\infty} \int_{-\infty}^{\infty} \Phi(tx) f(x)\, dx = \frac{1}{p} \int_0^p \Phi(x)\, dx \int_{-\infty}^{\infty} f(x)\, dx. \tag{2.12}$$

Proof: In view of Theorem 2.1 it suffices to prove (2.12) for the case $t \to -\infty$, and this is easily done.

The proof of the n-dimensional Riemann-Lebesgue lemma using Corollary 2.2 is straightforward (See, for example, [7], p. 316).

3. Some Consequences of the Main Result

Let $\mathcal{B}(R^n)$ denote the σ-algebra of all Borel subsets of R^n and let $\mathcal{A}$ be any σ-algebra of Lebesgue measurable subsets of R^n with $\mathcal{B}(R^n) \subseteq \mathcal{A}$. Then we have:

COROLLARY 3.1: *Let $\mu: \mathcal{A} \to R$ be a measure with $\mu(R^n) = 1$ and $d\mu << dx$. If Φ is a bounded n–tuply periodic function on R^n and if $\Phi \in L^1(R^n, d\mu)$, then*

$$\lim_{m(t) \to \infty} \int_{R^n} \Phi_t(x)\, d\mu(x) = \frac{1}{|B|} \int_B \Phi(x)\, dx. \tag{3.1}$$

Proof: Since $d\mu << dx$, then it follows from the Lebesgue-Radon-Nikodým theorem [6] that there exists a nonnegative $\mathcal{A}$-measurable function f on R^n such that

$$\int_{R^n} \Phi_t(x)\, d\mu(x) = \int_{R^n} \Phi_t(x) f(x)\, dx \tag{3.2}$$

and

$$\mu(A) = \int_A f(x)\, dx,\ A \in \mathcal{A}. \tag{3.3}$$

In particular

$$\int_{R^n} f(x)\, dx = \mu(R^n) = 1$$

implies $f \in L^1(R^n, dx)$ and the result follows from (3.2) and Theorem 2.1.

We can put this last result in a probabilistic framework as follows: Let $(\Omega, \mathfrak{F}, P)$ be a probability space i.e., Ω is any set, $\mathfrak{F}$ is a σ-algebra of subsets of Ω and P is a measure on $\mathfrak{F}$ with $P(\Omega) = 1$. Then it is well known [1] that if

$$X = (X_1, \cdots, X_n): \Omega \to R^n$$

is a "random vector" i.e., each $X_i: \Omega \to R$, $1 \leq i \leq n$ is an $\mathfrak{F}$-measurable function, there is a probability measure P_X on $\mathcal{B}(R^n)$ induced by the random vector X given by

$$P_X(A) = P\{\omega \epsilon \Omega \mid X(\omega) \epsilon A\},\ A \epsilon \mathcal{B}(R^n).$$

If $X: \Omega \to R^n$ is a random vector, $g: R^n \to R$ is a Borel measurable function and if

$$E(g \circ X) = \int_\Omega (g \circ X)(\omega)\, dP(\omega)$$

is the "expected value" of $g \circ X$, then

$$E(g \circ X) = \int_{R^n} g(x)\, dP_X(x). \tag{3.4}$$

In the sense that if either integral exists, so does the other, and the two are equal.

It is said that a random vector X on Ω has a "density" if $dP_X << dx$. In this case we see, as in the proof of Corollary 3.1, that there is a nonnegative, Borel measurable function f on R^n such that $dP_X = f(x)\, dx$. We call f the "density function" of X.

Using these facts we can now prove the following:

COROLLARY 3.2: *Let $(\Omega, \mathfrak{F}, P)$ be a probability space and let $X: \Omega \to R^n$ be a random vector with density. If Φ is a bounded Borel measurable, n-tuply periodic function and if $\Phi \in L^1(R^n, dP_X)$, then*

$$\lim_{m(t)\to\infty} E(\Phi \circ X_t) = \frac{1}{|B|} \int_B \Phi(x)\, dx, \tag{3.5}$$

where $X_t = (t_1 X_1, \cdots, t_n X_n)$.

Proof: From (3.4) we have

$$E(\Phi \circ X_t) = \int_{R^n} \Phi_t(x)\, dP_X(x).$$

The result follows from Corollary 3.1.

INSTITUTO DE INVESTIGACIONES
EN MATEMÁTICAS APLICADAS Y EN SISTEMAS
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

REFERENCES

[1] R. B. ASH, Real Analysis and Probability, Academic Press, New York, 1972.
[2] N. K. BARY, A Tratise on Trigonometric Series, Pergamon Press, New York, 1964.

[3] N. BOURBAKI, Elements of Mathematics, General Topology. Part 2. Addison-Wesley Pub. Co., Reading, Massachusetts, 1966.

[4] L. FEJÉR, *Untersuchungen über Fourierische Reihen*. Math. Ann. **58** (1904), 501–09.

[5] W. H. FLEMING, Functions of Several Variables. Addison-Wesley Pub. Co., Reading, Massachusetts, 1965.

[6] E. HEWITT AND K. STROMBERG. Real and Abstract Analysis. Springer Verlag, New York, 1965.

[7] T. KAWATA. Fourier Analysis in Probability Theory. Academic Press, New York, 1972.