s-S SPACES

By Gerard P. Protomastro

1. Introduction

A projection in a Banach space is any linear operator P in the space with the property that $P^2 = P$. One of the main problems connected with the notion of projections in Banach spaces is the following: Given a Banach space X, under what conditions does there exist a projection onto X of any space $Z \supset X$? Moreover, what may be said of the norms of such projections? Two important concepts in this connection are the P and S spaces. A Banach space X is said to be a P space if for every Banach space Z containing X there exists a projection P from Z onto X; moreover, if P can always be taken so that $\| P \| \leq \lambda$, we say that X is a P_λ space. If in the above we consider X and Z to be separable Banach spaces, then X is called a S space and S_λ space, respectively.

In this present paper we consider a class of spaces, namely s-S spaces, which properly contains the class of S spaces. A separable Banach space X is said to be a *s-S space* if for every separable Banach space Z containing X there is a subspace X_1 of X such that X_1 is isomorphic to X and X_1 is complemented in Z (i.e., there exists a projection P from Z onto X_1); moreover, if P can always be taken so that $\| P \| \leq \lambda$, we say that X is a s-S_λ space. In section 2 we investigate some properties of s-S spaces. In section 3 we show that (c), the space of convergent sequences, is a s-S_1 space. We conclude the paper with some open problems.

All Banach spaces and subspaces under discussion are assumed to be infinite dimensional. All linear operators are assumed to be bounded.

2. Properties of s-S Spaces

Theorem 2.1. *If X is a s-S space and X is isomorphic to Y, then Y is a s -S space.*

Proof. Let Z be a separable Banach space containing Y and let i be an isomorphism from X onto Y. By [4, p. 166] there exists a separable Banach space Z_1 containing X and an isomorphic extension I of i from Z_1 onto Z. Since X is a s-S space there is a subspace X_1 of X such that X_1 is isomorphic to X and X_1 is complemented in Z. Let P_{X_1} be the projection from Z onto X_1. The subspace $Y_1 = i(X_1)$ of Y is isomorphic to Y and Y_1 is complemented in Z since $P_{Y_1} = iP_{X_1}I^{-1}$ is a projection from Z onto Y_1. Hence Y is a s-S space.

Corollary 2.2. *If X is isometrically isomorphic to Y and X is a s-S_λ space, then Y is a s-S_λ space.*

Proof. In the proof of theorem 2.1 we can choose P_{X_1} such that $\| P_{X_1} \| \leq \lambda$. Thus $\| P_{Y_1} \| = \| iP_{X_1}I^{-1} \| \leq \lambda$.

THEOREM 2.3. *If X is a s-S space, then*
(1) *X contains a subspace isomorphic to* (c), *and*
(2) *X cannot be isomorphic to a conjugate space of a Banach space.*

Proof. (1) Since X is separable we may embed it onto a subspace Y of $C[0, 1]$. By theorem 2.1 Y is a s-S space, and thus there exists a subspace Y_1 of Y complemented in $C[0, 1]$. It follows by [2, p. 221] that Y_1 must contain a subspace isomorphic to (c). Thus Y, and hence X, must contain a subspace isomorphic to (c).

(2) Suppose that X is isomorphic to a conjugate space Z^* of a Banach space Z. In part (1) Y_1 is isomorphic to X and thus Y_1 is also isomorphic to Z^*. It follows by [2, p. 221] that Y_1, and hence X, must contain a subspace isomorphic to (m), where (m) is the space of bounded sequences. But since X is separable this is impossible.

COROLLARY 2.4. *ℓ_p and $L_p[0, 1]$, $1 \leq p < \infty$, are not s-S spaces.*

Conditions (1) and (2) of theorem 2.3 are necessary, but not sufficient, conditions of s-S spaces. To show that condition (1) is not sufficient we consider a subspace (ℓ) of (c) constructed by Sobczyk in [5, p. 84]. (ℓ) has the property that it is not complemented in (c), and thus it is not isomorphic to (c) by [2, p. 217]. (ℓ) contains a subspace isomorphic to (c), since every infinite dimensional subspace of (c) contains a subspace isomorphic to (c). If (ℓ) is a s-S space, then there must be a subspace X of (ℓ) such that X is isomorphic to (ℓ) and complemented in (c). This implies that X, and hence (ℓ), is isomorphic to (c), which is impossible. Thus (ℓ) is not a s-S space. The space $L_1[0, 1]$, which is not a s-S space by corollary 2.4, shows us that condition (2) is not sufficient.

Definition 2.5. A linear transformation $T : X \to Y$ is said to be of *Sobczyk type II-A* if the null space $N(T)$ of T has a closed complement N^c, and if T restricted to N^c is an isomorphism of N^c onto the range $R(T)$ of T.

THEOREM 2.6. *If X is a s-S space, then for every separable Banach space* Z *containing X there exists a linear transformation T of Sobczyk type* II-A *from* Z *onto X.*

Proof. Since X is a s-S space, there exists a subspace Y of X such that Y is isomorphic to X, say under I, and Y is complemented in Z. Let P be a projection from Z onto Y. We define $T = IP$. We note $N(T) = N(P)$, so that $N(T)$ has a closed complement, namely Y. Furthermore, $T|_Y = I$ is an isomorphism of Y onto $R(T) = X$. Thus T is a linear transformation of Sobczyk type II-A from Z onto X.

3. s-S_1 Spaces

It is known that the class of S_1 spaces is empty. The next theorem shows that this is not the case with the class of s-S_1 spaces.

THEOREM 3.1. *(c) is a s-S_1 space.*

Proof. Let Z be a separable Banach space containing (c). For each i, we define $d_i(x) = x_i$ for every $x \in (c)$, and note each d_i is a linear functional on (c) with $\| d_i \| = 1$. Then it follows by the Hahn-Banach Theorem that each d_i can be extended to a linear functional z_i^* on Z such that $\| z_i^* \| = \| d_i \| = 1$. Since Z is separable there exists a subsequence $\{z_{n_i}^*\}$ of $\{z_i^*\}$ which converges in the w^*-topology, i.e. $\{z_{n_i}^*\}$ is pointwise convergent, by [7, p. 209]. We then define X_1 to be the subspace of (c) of sequences x for which $x_{n_i} = x_{n_i+1} = \cdots = x_{n_{i+1}-1}$ for every i and $x_n = 0$ for $n < n_1$. We also define $P : \mathrm{Z} \to X_1$ by $P : z \to Pz$ where $(Pz)_n = z_{n_i}^*(z)$, if $n_i \leq n < n_{i+1}$, and $(Pz)_n = 0$, if $n < n_1$. It is clear that $P(\mathrm{Z}) \subset X_1$. If $x \in X_1$, then $(Px)_n = z_{n_i}^*(x) = d_{n_i}(x) = x_{n_i} = x_n$, for $n_i \leq n < n_{i+1}$, and $(Px)_n = 0 = x_n$, for $n < n_1$. Thus $Px = x$ for every $x \in X_1$. Furthermore, for $z \in \mathrm{Z}$

$$\| Pz \| = \sup_n | (Pz)_n | \leq \| z_{n_i}^* \| \| z \| = \| z \|.$$

Hence P is a projection from Z onto X_1 such that $\| P \| = 1$.

It remains to show that X_1 is isomorphic to (c). We define $I : (c) \to X_1$ by $I : x \to Ix$ where $(Ix)_n = x_i$, if $n_i \leq n < n_{i+1}$, and $(Ix)_n = 0$, if $n < n_1$. Clearly $I(c) \subset X_1$ and we claim that I is onto. For let $x \in X_1$ and let $y \in (c)$ be defined by $y_i = x_{n_i}$. Then

$$(Iy)_n = y_i = x_{n_i} = x_n, \quad \text{for} \quad n_i \leq n < n_{i+1},$$

and

$$(Iy)_n = 0 = x_n, \quad \text{for} \quad n < n_1.$$

Thus $Iy = x$ and I is onto as claimed. Finally we note for every $x \in (c)$

$$\| Ix \| = \sup_n | (Ix)_n | = \sup_i | x_i | = \| x \|,$$

i.e. $\| Ix \| = \| x \|$. Thus X_1 is isomorphic to (c), and hence (c) is a s-S_1 space.

4. Open Problems

Definition 4.1. A set $\{x_n\}$ of elements of a Banach space X is called an *unconditional basis* of X if for every $x \in X$ there is a unique sequence of reals $\{a_n\}$ such that $x = \sum_1^\infty a_n x_n$ and this series converges unconditionally.

It is not hard to show, using results from [3, p. 100] and [1, p. 295], that a separable Banach space X with an unconditional basis is a s-S space if and only if X is isomorphic to a $C(H)$ space, where H is a compact metric space.

Problem 1. Let X be a separable Banach space. Is it true that X is a s-S space if and only if X is isomorphic to a $C(H)$ space, where H is a compact metric space?

If the above is not true, we could state the following general problem.

Problem 2. Characterize the s-S spaces.

The class of P_1 spaces has been completely characterized as those spaces which are isometrically isomorphic to a $C(K)$ space, where K is compact and extremally disconnected. We have stated in section 3 that the class of S_1 spaces is empty.

Problem 3. Characterize the s-S_1 spaces.

Let $s[X] = \inf \{\lambda : X \text{ is a } s\text{-}S_\lambda \text{ space}\}$. Theorem 3.1 implies that $s[(c)] = 1$. Using a result from [6, p. 942], it follows that $s[(c_0)] \leq 2$, where (c_0) is the space of sequences which converge to 0.

Problem 4. Is it true that $s[(c_0)] = 2$? If $s[(c_0)] \neq 2$, then what is the value of $s[(c_0)]$?

We conclude this section by considering a generalization of s-S spaces.

Definition 4.2. The *density character* $\delta(X)$ of a Banach space X is defined as the smallest cardinal number such that there exists a dense subset of X having that cardinal number.

Definition 4.3. A Banach space X of density character Γ is said to be a s-$S(\Gamma)$ *space* if for every Banach space Z containing X, where $\delta(Z) = \Gamma$, there is a subspace X_1 of X such that X_1 is isomorphic to X and X_1 is complemented in Z.

Problem 5. Characterize the s-$S(\Gamma)$ spaces.

St. Peter's College
Jersey City, New Jersey

References

[1] J. Lindenstrauss and A. Pelczynski, *Absolutely summing operators in L-spaces and their applications*, Studia Math. **29** (1968), 275–326.

[2] A. Pelczynski, *Projections in certain Banach spaces*, Studia Math. **19** (1960), 209–28.

[3] Z. Semadeni, *Isomorphic properties of Banach spaces of continuous functions*, Studia Math., Serie Specjalna **1** (1963), 93–108.

[4] A. Sobczyk, *On the extension of linear transformations*, Trans. Amer. Math. Soc. **55** (1944), 153–69.

[5] A. Sobczyk, *Projections in Minkowski and Banach spaces*, Duke Math. J. **8** (1941), 78–106.

[6] A. Sobczyk, *Projection of the space (m) on its subspace (c_0)*, Bull. Amer. Math. Soc. **47** (1941), 938–47.

[7] A. Taylor, Introduction to functional analysis, Wiley, New York, 1958.