A PROPERTY OF BAIRE FIRST CATEGORY SETS

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Abstract. In this paper it is shown that every Baire first category set misses a Cantor set in every interval and an example is given to show that the converse does not hold.

In what follows every set which is mentioned is a subset of the set of all real numbers.

Let us recall that a set is called *nowhere dense* if and only if it misses an interval in every interval. Moreover, a set is called *Baire first category* (or simply, *first category*) if and only if it is a countable union of nowhere dense sets. Thus, it is reasonable to expect that a first category set would also miss infinitely many points in every interval. Indeed, a first category set misses continuumly many points in every interval since (as shown in the Theorem below) a first category set misses a *Cantor ternary set* (or simply, a *Cantor set*) in every interval. However, as also shown below, the property of missing continuumly many points in every interval does not characterize a set of first category.

Theorem 1. *Let F be a set of first category and I an interval. Then there exists a Cantor set C such that $C \subseteq (I - F)$.*

Proof. From the definition of a first category set it follows that

(1) $F = N_0 \cup N_1 \cup N_2 \cup \cdots \cup N_i \cup \cdots$ *with* $i \in \omega$

where N_i is a nowhere dense set for every $i \in \omega$.

Since a nowhere dense set misses an interval in every interval, it is clear that every nowhere dense set misses two disjoint closed intervals in every closed interval.

Let C_0 be a closed interval such that $C_0 \subseteq I$. In view of the above, N_0 misses two disjoint closed subintervals C_{00} and C_{01} of C_0. Similarly, N_1 misses two disjoint closed subintervals C_{000} and C_{001} of C_{00}, and, N_1 misses two disjoint closed subintervals C_{010} and C_{011} of C_{01}. Again, N_2 misses two disjoint closed subintervals C_{0000} and C_{0001} of C_{000}, and, N_2 misses two disjoint closed subintervals C_{0010} and C_{0011} of C_{001}, and, N_2 misses two disjoint closed subintervals C_{0100} and C_{0101} of C_{010}, and, N_2 misses two disjoint closed subintervals C_{0110} and C_{0111} of C_{011}. Continuing in this way, we see that N_{i+1} misses 2^{i+2} pairwise disjoint closed subintervals of the 2^{i+1} closed intervals which are missed by N_i. But then from (1) it follows that the intersection K of all these closed subintervals is missed by F. Thus, $K \subseteq (I - F)$. However, it is obvious that $C \subseteq K$ for some Cantor set C. Hence the Theorem is proved.

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COROLLARY (Baire category theorem). *No set containing an interval is of first category.*

Proof. Let S be a set such that $I \subseteq S$ for some interval I. If S were of first category then Theorem 1 would imply $I \nsubseteq S$ which would contradict $I \subseteq S$.

Next, we prove that the converse of Theorem 1 does not hold.

THEOREM 2. *There exists a set A such that A misses a Cantor set in every interval and such that A is not of first category.*

Proof. Let I be an interval. By ([1], Theorem 1.6) the interval I is a disjoint union of a set F of first category and a set A of zero Lebesgue measure. Clearly, A is not of first category (because otherwise, $F \cup A = I$ would be of first category contradicting the Corollary above). On the other hand, A being of zero Lebesgue measure misses a closed set of positive Lebesgue measure in every interval. Thus, a priori A misses a Cantor set in every interval. Consequently, A is a set which satisfies the conclusion of the Theorem.

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REFERENCE

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