THE SPUN SQUARE KNOT IS THE SPUN GRANNY KNOT

By D. Roseman

In this paper a knot will be a particular embedding of S^n in R^{n+2} for some $n > 0$; this embedding may be thought of as either in the smooth category or the category of PL manifolds where we will consider only locally unknotted embeddings.

The principal result of this paper is to show that the spun square knot is the same (in the sense of ambient isotopy) as the spun square knot. Generalizations of this result will also be considered.

The knotted 2-sphere in 4-space called the spinning of a knot was first considered by Artin [1]. We answer the question: If K and K' are distinct knots, are the corresponding spun knots $S(K)$ and $S(K')$ distinct? The answer will be negative.

We will first show that the spun right hand trefoil is ambiently isotopic to the spun left hand trefoil. This result is not too satisfying, since although the right-hand trefoil is not ambiently isotopic to the left-hand trefoil (Dehn [2] or footnote in Fox [3]) there is a (orientation reversing) homeomorphism of R^3 (a reflection about a plane) such that the image of the right-hand trefoil is the left-hand trefoil. We will then show a stronger result; that the spun square knot is ambiently isotopic to the spun granny knot; in [9] and [3] it is shown that there is no homeomorphism of R^3 onto itself which takes the square knot onto the granny knot.

By the isotopy extension theorems the notion of ambient isotopy is shown to coincide with that of (in the PL case, locally unknotted) isotopy of the knot. We will explicitly show that certain knots are equivalent by describing the projection, from R^4 to R^3, of the locally unknotted isotopy from one embedding to the other. The notion of the projection of a knot is discussed in [11] and [6].

If $K: S^2 \to R^4$ is a knot, the projection K^* of the knot is defined by $K^* = \pi \circ K(S^2)$ where $\pi: R^4 \to R^3$ is defined to be the projection $p(x, y, z, w) = (x, y, z, 0)$. If $p \in S^2$ and $K(p) = (x_p, y_p, z_p, w_p)$, the height of p, or the height of $K(p)$, will refer to the number w_p. The double point set of the projection will be the set of points in the projection whose preimage, under $(\pi \circ K)^{-1}$, contains more than one point.

There are two important aspects of a knot that are difficult to display when we draw a picture, on a page, of the projection of a knotted 2-sphere. Firstly, there are hidden surfaces, that is, portions of the projection which, from the reader's viewpoint pass behind others. The other difficulty is that the height relations at points corresponding to the double point set are not explicit.

In the cases of the spun knots we will consider, this first will cause little difficulty since it is possible to keep track of the hidden surfaces by taking a projection so that the projection of the spun knot is the spinning of the projec-

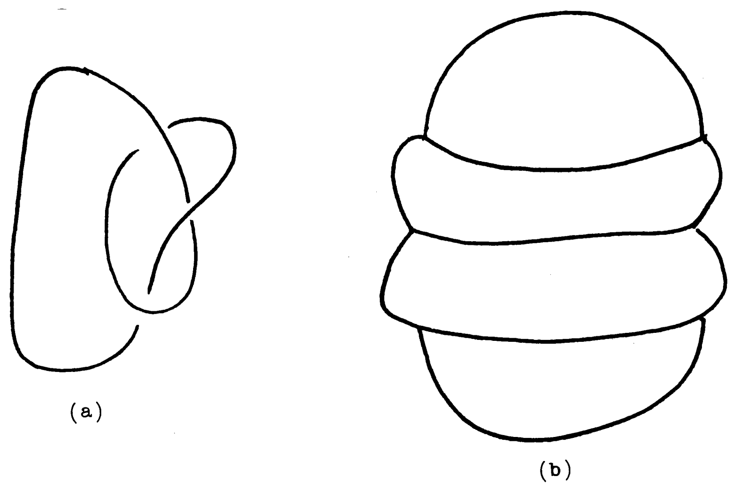


Fig. 1

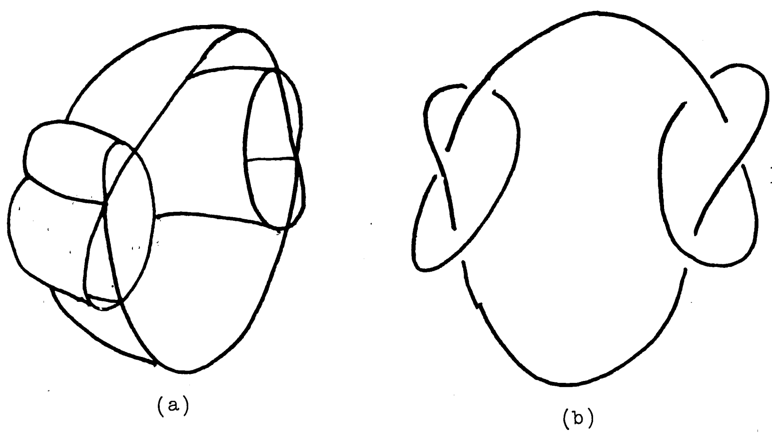
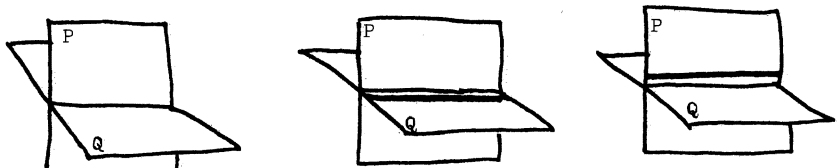


Fig. 2

tion of the knot [7]. If, for example, we spin the knotted arc (that associated with a trefoil knot) Figure 1a, we may draw the spun knot as in Figure 1b.

We can illustrate the hidden surfaces by cutting the projection in half by a plane, Figure 2b. We remark, in passing, that this plane which slices the projection in half can be thought of as a projection of a 3-dimensional hyperplane in R^4 whose intersection with the knot in R^4 is the knotted circle whose projection lies in this plane. In this case the knot can be seen to be a square knot. Furthermore, by taking other hyperplanes in R^4 parallel to this one, one may obtain other slice knots and links and verify that Example 9 of Fox [4] is a spun trefoil knot.

One graphical method for indicating height relationships in projections of knots is the following. Suppose that a portion of a projection looks like two 2-disks intersecting transversely in R^3, Figure 3a. These two disks are projections of disjoint disks in R^4, one above the other in R^4 and we may say that the double



(a) (b) Disk P lies above disk Q (c) Disk P lies below disk Q

Fig. 3

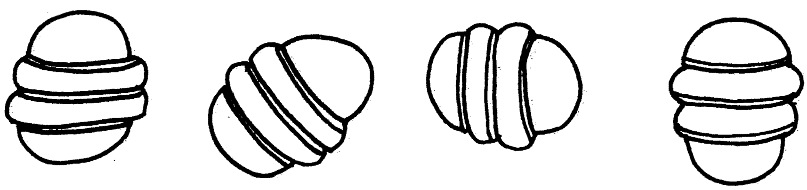
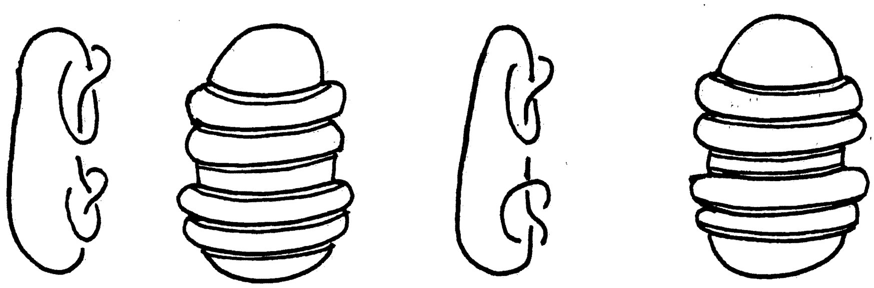


Fig. 4



granny spun granny square spun square

Fig. 5

point set is the shadow of the higher disk upon the lower disk (if we had a light source whose rays are parallel to the last coordinate of R^4). If we change the direction of the light source slightly we will find that the shadow will fall on the lower disk near the double point set (Figure 3b and 3c).

Using these conventions we can see how the sequence of drawings in Figure 4 is the projection of an ambient isotopy from the spun right hand trefoil to the spun left hand trefoil; this ambient isotopy is in fact a rotation of R^4.

We next wish to show that the spun square knot is equivalent to the spun granny knot, Figure 5.

The projection of the locally flat isotopy between these two knots is given in Figures 6_1 through 6_{12}. Here in these drawings, we omit the drawing of the shadows as in Figure 4, these may be filled in by the reader. It will not matter whether one thinks of Figure 6_1 as the projection of the spun square knot or the spun granny; in either case, the projections if interpreted consistently will

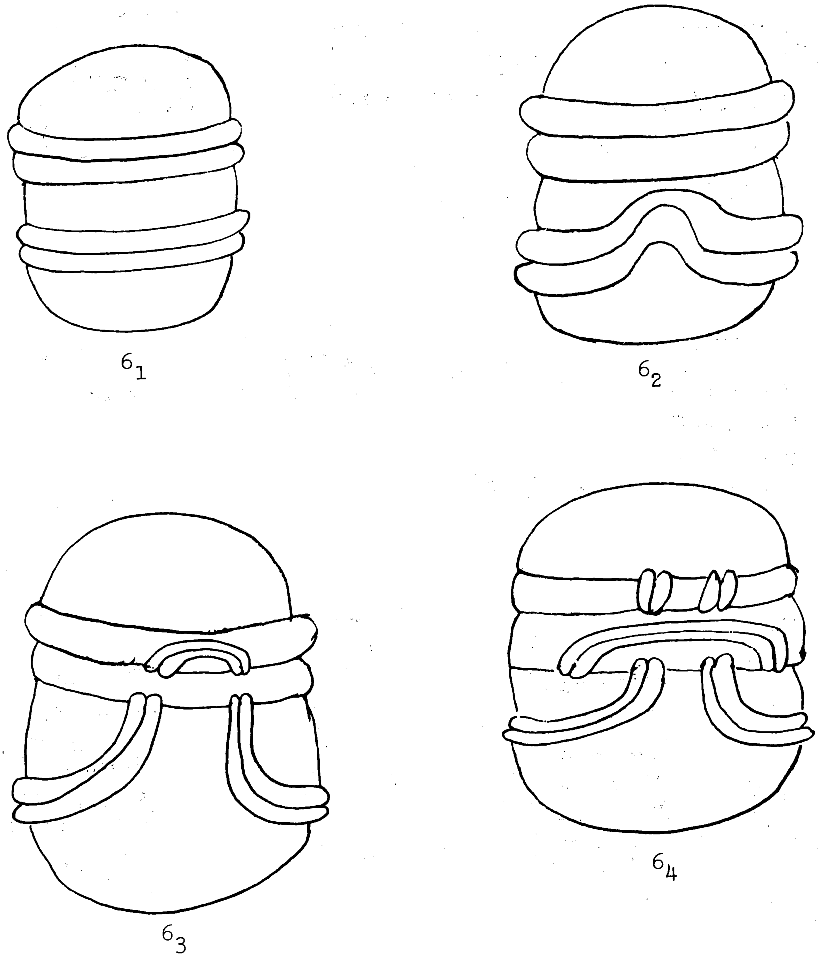


Fig. 6

describe the desired isotopy from one to the other. In Figure 6 it is important to keep careful track of the height relations in order to make sure that the regular homotopy of the projection shown is the projection of an isotopy in R^4; recall that Smale has shown [10] that any immersion of S^2 in R^3 is regularly homotopic to the standard embedding. The isotopy projected in Figure 6 is a special case of a more general construction which is defined in [8].

The knot projected in Figure 6_3 may also be described as follows. When we produce a spun knot, we think of taking a rigid knotted arc and spinning it with the endpoints fixed. If we allow the arc to flex and move as we spin it, as long as it returns to its original position by the time we finish spinning, we may obtain a different sort of knot called a deform-spun sphere, [5]. If the arc moves as in Figure 7, then the projection of the corresponding deform-spun sphere (equivalent to the spun sphere) will be 6_3.

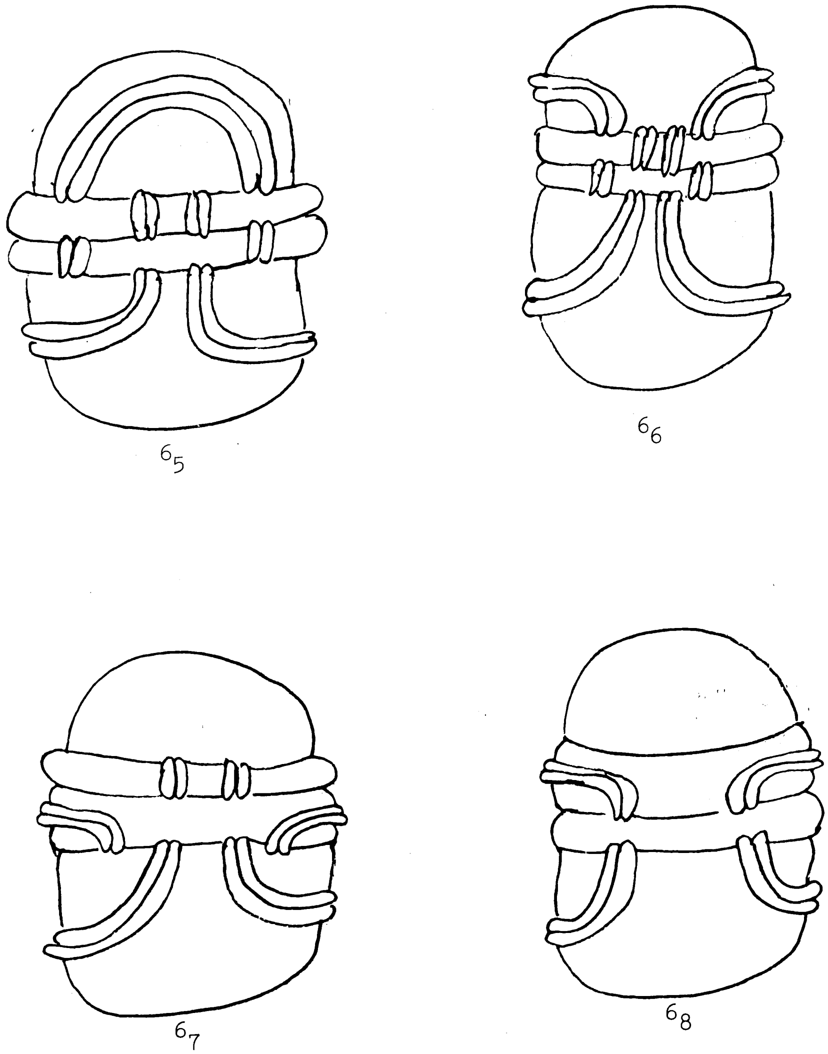


Fig. 6

To get from 6_3 to 6_4, we may view 6_3 in a slightly different way; we think of 6_3 as being obtained by taking a version of the spun trefoil whose projection looks like Figure 8_1, then tying this like a spun trefoil, then 6_4 is obtained by taking a version of the spun trefoil whose projection looks like Figure 8_2, then tying this like a spun trefoil.

We note that the isotopy shown in Figure 6 is a higher dimensional analogue of the isotopy which shows commutativity of addition of knots [4].

One can generalize this result to higher dimensions and show:

If K is a knot (of whatever dimension) and $-K$ is its mirror image, then the spinning of $K \# -K$ is ambiently isotopic to the spinning of $K \# K$. Here #

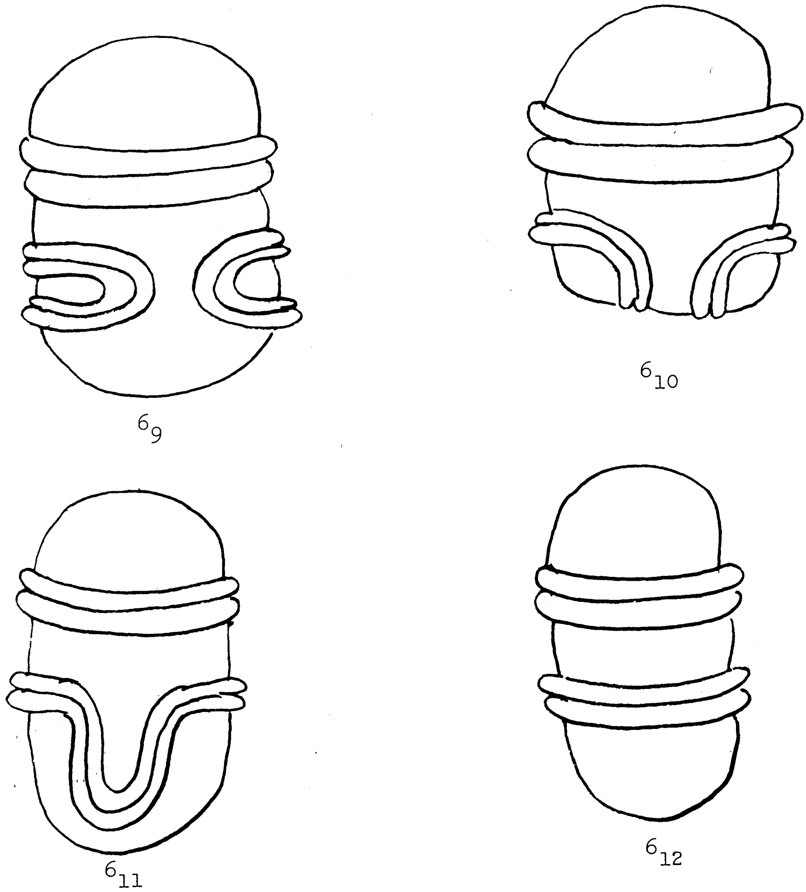


Fig. 6

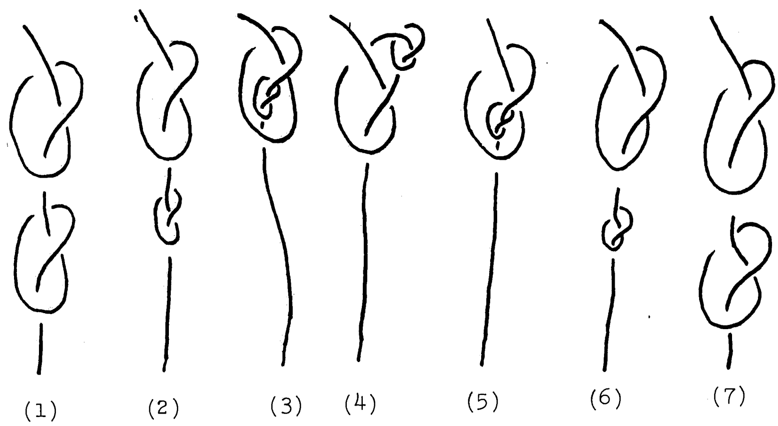
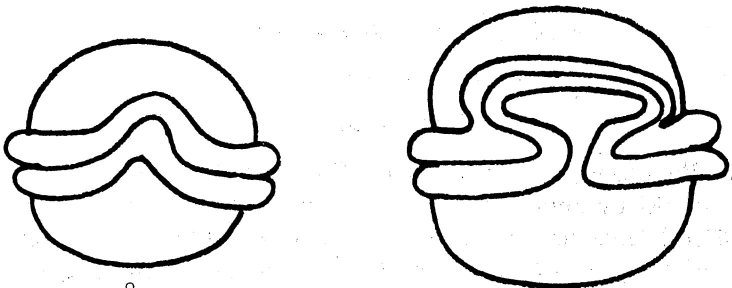


Fig. 7



8_1 8_2

Fig. 8

denotes connected sum of knots, and the spinning referred to can be a more generalized type of spinning. Proof of this general result will be found in [8].

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