A MOD p WU FORMULA

By Franklin P. Peterson[1]

§1. Introduction

In 1950, Wu Wen-tsen [7] showed how the Steenrod algebra acts on the mod 2 cohomology of BO by giving an expression for $Sq^t(W_n)$. The corresponding formulae for $\mathcal{P}^t(c_n)$ have not yet appeared. Mukohda and Sawaki [5] computed the coefficient of $c_{n+t(p-1)}$ in $\mathcal{P}^t(c_n)$ to be $\binom{n-1}{t}$. Borel and Serre [1] gave some low dimensional formulae. Llorente [3] gives a formula for $p = 3$. Recently, B. Shay [6] has given a complete formula for $\mathcal{P}^t(c_n)$ in terms of monomials in the Chern classes. The formula is rather complicated and calculating the coefficients even for small values of t does not seem easy.

In this note, we give formulae for $\mathcal{P}^t(c_n)$ not in terms of monomials in the Chern classes but in terms of Chern classes and particular symmetric functions. We hope that these formulae will be reasonably easy to compute with.

Our method is the same as the above mentioned authors, namely, to consider $H^*(BU)$ as symmetric polynomials on two dimensional classes and use our knowledge of what $\mathcal{P}^t$ does to two dimensional classes.

§2. Statement of Results

Let ω be a partition of n. Let s_ω be the corresponding symmetric function in two dimensional generators, so that $s_\omega \in H^{2n}(BU; Z_p)$, where p is an odd prime. For example, $c_n = s_{(1,1,\cdots,1)}$. Our main result is the following theorem.

Theorem 2.1.

$$\sum_{i=0}^{t(p-2)} (-1)^i c_{n+i}\Big(\sum_{\substack{1\le j_1\le\cdots\le j_t\le p-1\\ \Sigma_k j_k=t(p-1)-i}} s_{(j_i,\cdots,j_t)}\Big) =$$

$$\sum_{k=0}^{t} (-1)^k \binom{n+kp-t}{k} \mathcal{P}^{t-k}(c_{n+k(p-1)}).$$

One can solve for $\mathcal{P}^t(c_n)$ inductively using 2.1. This is done in the following theorem.

Theorem 2.2

$$\mathcal{P}^t(c_n) = \sum_{i=0}^{t(p-2)} (-1)^i c_{n+i} \sum_{k=0}^{t-1} \alpha_{k,n,t} \Big(\sum_{\substack{1\le j\le p-1\\ j=t(p-1)-i}} s_{(j_t,\cdots,j_{t-k})}\Big) +$$

$\binom{n-1}{t} c_{n+t(p-1)}$, *where* $\alpha_{0,n,t} = 1$ *and* $\alpha_{k,n,t} = (-1)^{k+1} \binom{n-t+kp}{k} + (-1)^k \sum_{\ell=1}^{k-1} \binom{n-t+(k-\ell)p}{k-\ell} \binom{n+2\ell p-t}{\ell}$ *if* $k > 0$.

This formula reduces, for $t = 1$, to the following corollary.

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COROLLARY 2.3. *$\mathcal{P}^1(c_n) = (n-1)c_{n+p-1} + \sum_{i=0}^{p-2} (-1)^i c_{n+i} s_{(p-1-i)}$, where $s_{(i)}$ is the primitive element in $H^{2i}(BU; Z_p)$.*

For $p = 3$, we need only consider s_ω where ω has ones and twos. We now show how to express such s_ω as polynomials in the Chern classes. Let $\not{s}_R = s_\omega$, where $R = (r_1, \cdots)$ is given by $r_1 =$ the number of ones in ω, $\cdots$, $r_j =$ the number of j's in ω.

THEOREM 2.4 $\not{s}_{(r_1,r_2)} = c_{r_1+r_2} \cdot c_{r_2} + \sum_{k=1}^{r_2} \beta_{k,r_1} c_{r_1+r_2+k} \cdot c_{r_2-k}$,

where

$$\beta_{k,r_1} = -\binom{r_1+2k}{k} + \sum_{\ell=1}^{k-1} \binom{r_1+2\ell}{\ell}\binom{r_1+2k}{k-\ell} = (-1)^k \frac{r_1+2k}{r_1+k}\binom{r_1+k}{k}.$$

§3. Proofs

The proof of the main result, theorem 2.1, is based on the following well-known proposition which shows how to multiply $s_\omega \cdot s_{\omega'}$. Let X range over all infinite matrices (x_{ij}), with $i \geq 0$, $j \geq 0$, but x_{00} omitted. Let $r_i = \sum_j x_{ij}$, $s_j = \sum_i x_{ij}$, $t_n = \sum_{i+j=n} x_{ij}$, and define $b(X) =$ the multinomial coefficient $\Pi\, t_n!/\Pi\, x_{ij}!$. Let $R(X) = (r_1, \cdots)$, $S(X) = (s_1, \cdots)$, and $T(X) = (t_1, \cdots)$.

PROPOSITION 3.1. $\not{s}_R \cdot \not{s}_S = \sum_{S(X)=S}^{R(X)=R} b(X) \not{s}_{T(X)}$.

We remark that this is similar to the multiplication of Milnor basis elements in the Steenrod algebra except that the row sum is not weighted [4]. A proof along the same line can be given.

We now prove theorem 2.1. Consider

$$\sum_{i=0}^{t(p-2)} (-1)^i c_{n+i} \Big(\sum_{\substack{1 \leq j_1 \leq \cdots \leq j_t \leq p-1 \\ \Sigma_k j_k = t(p-1)-i}} s_{(j_1,\cdots,j_t)} \Big).$$

Consider the coefficient of $\not{s}_{(r_1,r_2,\cdots,r_{p-1},r_p)}$ in this sum, where $\sum k r_k = n + t(p-1)$. Using 3.1 we see that this term can occur as a product

$$\not{s}_{(e)} \cdot \not{s}_{(r_1-\epsilon_1+\epsilon_2, r_2-\epsilon_2+\epsilon_3, \cdots, r_k-\epsilon_k+\epsilon_{k+1}, \cdots, r_{p-1}-\epsilon_{p-1}+r_p, 0)},$$

where

$0 \leq \epsilon_k \leq r_k$ for $1 \leq k \leq p-1$. $\sum_{k=1}^{p-1} k(r_k - \epsilon_k + \epsilon_{k+1})$

$$= \sum_{k=1}^{p-1} k r_k - \Big(\sum_{k=1}^{p-1} \epsilon_k\Big) + (p-1) r_p - \epsilon_1.$$

Setting $\epsilon_p = r_p$, we obtain $\sum_{k=1}^{p-1} k(r_k - \epsilon_k + \epsilon_{k+1}) = n + t(p-1) - e$, where $e = \sum_{k=1}^{p} \epsilon_k$. Also, we must have $\sum_{k=1}^{p-1} (r_k - \epsilon_k + \epsilon_{k+1}) = t$, hence $\sum_{k=1}^{p} r_k = t + \epsilon_1$ and ϵ_1 is determined by the other data. Hence, the coefficient of $\not{s}_{(r_1,\cdots,r_p)}$ is $\sum_{\substack{\epsilon_k \leq r_k \\ k=2,\ldots,p-1}} \Pi_{k=1}^{p-1} \binom{r_k}{\epsilon_k} (-1)^{e-n}$. If some $r_k > 0$ for $k = 2, \cdots, p-1$, then we can hold ϵ_j fixed for $j \neq k$ and sum on ϵ_k to see that this coefficient is zero because $\sum_{\epsilon=0}^{r} (-1)^{\epsilon}\binom{r}{\epsilon} = 0$. If $r_k = 0$ for $k = 2, \cdots, p-1$, then the coefficient of $\not{s}_{(n-t+kp,0,\cdots,0,t-k)} = \mathcal{P}^{t-k}(c_{n+k(p-1)})$ is $(-1)^{e-n}\binom{r_1}{\epsilon_1} = (-1)^{k(p-2)}\binom{n-t+kp}{n+k(p-1)-t}$ $= (-1)^k\binom{n-j+kp}{k}$.

This proves 2.1.

The proof of 2.2 from 2.1 is an easy exercise and is left to the reader. The coefficient of $c_{n+t(p-1)}$ is $\binom{n-1}{t}$ by the results of [5].

The first part of theorem 2.4 follows easily from the following proposition which in turn is an immediate corollary of 3.1.

PROPOSITION 3.2. $c_{r_1+r_2} \cdot c_{r_2} = \sum_{k=0}^{r_2} \binom{r_1+2k}{k} \not{s}_{(r_1+2k, r_2-k)}$.

The second part of 2.4 can be found in [2] (see also [3]).

We remark in closing that in an appropriate sense (see [6]), theorem 2.1 holds over the integers. Theorem 2.2 does also if the coefficient of $c_{n+t(p-1)}$ is replaced by $\binom{n-1}{t} + p\binom{n-1}{t-1}$.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Cambridge, Massachusetts

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