CONSTRUCTION OF SOME NORMED MAPS

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1. Introduction

Let R be the real field and R^p the standard vector space over R, whose elements are the ordered collections $x = (x_1, \cdots, x_p)$, where each $x_i \in R$. As usual, the norm $n(x)$ is defined by the quadratic form

$$n(x) = x_1^2 + \cdots + x_p^2.$$

A bilinear map $\phi: R^p \times R^q \to R^r$ is said to be a *normed map* if

$$n(xy) = n(x)n(y), \tag{1.1}$$

for all $x \in R^p$, $y \in R^q$, where $xy = \phi(x, y)$.

The product of the classical algebras of complex, quaternion and Cayley numbers, respectively, $R^2 \times R^2 \to R^2$, $R^4 \times R^4 \to R^4$ and $R^8 \times R^8 \to R^8$, are normed maps. One generalization of these is the family of the Hurwitz-Radon maps ([6], [8], [14])

$$R^{\rho(r)} \times R^r \to R^r, \tag{1.2}$$

for $r = 1, 2, \cdots$, where $\rho(r)$ is defined as follows. Write $r = 2^{4a+b}(2k + 1)$ with $0 \leq b \leq 3$, then $\rho(r) = 8a + 2^b$. As it is well known, the maps (1.2) are all normed maps, and for $r = 1, 2, 4, 8$ they agree with the classical algebras.

Another generalization different from (1.2) is obtained, if we proceed as follows. First, consider the algebra structure in further dimensions. In general, this implies that property (1.1) is lost. Then, take restrictions to subspaces where (1.1) holds. The purpose of this paper is to develop such method, which will be outlined in the next paragraphs.

The construction of the Cayley algebra by means of couples of quaternions, introduced by Dickson in [5], sets an iterative pattern known as the Cayley-Dickson process. This was used by Albert in [3] to construct a sequence of algebras, $R = \mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \cdots \mathfrak{A}_r \subset \cdots$, where each $\mathfrak{A}_r$ is of dimension 2^r over R, and has as elements the set of all ordered pairs (x, y), for x, y in $\mathfrak{A}_{r-1}$.

The first algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$ are the complex, quaternion and Cayley numbers. For $r > 2$, the $\mathfrak{A}_r$'s are not associative, however, it was later proved by Schafer in [15] that all of them are *flexible*, which means some sort of restrictive associativity (see (2.7)). We have the usual relation $n(x) = x\bar{x}$, where $\bar{x}$ is the conjugate of x, and the trace is defined by $t(x) = x + \bar{x}$.

The product of $\mathfrak{A}_r$ induces a bilinear product $R^m \times R^m \to R^m$, where $m = 2^r$. Let $x, y \in \mathfrak{A}_r$ with $x = (x_1, x_2)$, $y = (y_1, y_2)$, where x_1, x_2, y_1, y_2 are elements of $\mathfrak{A}_{r-1}$. In general, for $r > 3$, we have $n(xy) \neq n(x)n(y)$. Suppose that each of the terms $n(x_1y_1)$, $n(x_1y_2)$, $n(x_2y_1)$ and $n(x_2y_2)$ fulfills the condition (1.1). Then, using the flexibility property, we can prove that $n(xy)$ satisfies (1.1) if and only if

$$(1.3) \qquad t[x_1(y_1, x_2, y_2)] = 0,$$

where $(y_1, x_2, y_2) = (y_1x_2)y_2 - y_1(x_2y_2)$ is the associator.

We will illustrate how this is used. With the algebra $\mathfrak{A}_4$ we get a multiplication $R^{16} \times R^{16} \to R^{16}$. If $R^{10} \subset R^{16}$ is the subspace formed with the elements (x, u), where x is a Cayley number and u a complex number, then the restriction $\phi: R^{10} \times R^{10} \to R^{16}$ is a normed map. In fact, if $w_1 = (x_1, u_1)$ and $w_2 = (x_2, u_2)$ are elements of R^{10} as above, then the associator $(x_2, u_1, u_2) = 0$, hence (1.3) is satisfied and $\phi(w_1, w_2) = w_1w_2$ defines a normed map, as asserted.*

This paper is almost self-contained. At the beginning, the results are developed for an arbitrary field and after section 3, we assume that the field has characteristic different from two. In section 6 we set a list of particular normed maps ((6.2), (6.13)), derived from some results previously established in section 5. In section 7 we give some general formulas, obtained by iterating the constructive steps developed in section 5. In section 8 some open problems are presented and discussed. Finally, in the appendix, some identities are set for the associator, in order to give more transparent proves of some results of Albert and Schafer.

2. Algebras constructed by the Cayley-Dickson process

Let $\mathfrak{A}$ be an algebra with unity element 1, having dimension m over a field F, and suppose that $\mathfrak{A}$ has an *involution* (involutorial antiautomorphism), that is, a linear operator $x \to \bar{x}$ on $\mathfrak{A}$, where $\bar{x}$ is called the *conjugate* of x, satisfying

$$\overline{xy} = \bar{y}\bar{x}, \qquad \bar{\bar{x}} = x$$

for all x, y in $\mathfrak{A}$, and

$$(2.1) \qquad x + \bar{x} = t(x)1, \qquad x\bar{x} = \bar{x}x = n(x)1,$$

where the elements $t(x)$, $n(x)$ are in F for all x in $\mathfrak{A}$, and are called, respectively, the *trace* and the *norm* of x.

By the Cayley-Dickson process we construct, for a fixed $\theta \neq 0$ in F, an algebra $\mathfrak{B} = \mathfrak{A}(\theta)$ of dimension $2m$ over F, having as elements the set of all ordered pairs $w = (x, y)$, for x, y in $\mathfrak{A}$, with addition of pairs and multiplication by scalars defined componentwise, and with multiplication of pairs $w_1 = (x_1, y_1)$, $w_2 = (x_2, y_2)$, defined by

$$(2.2) \qquad w_1w_2 = (x_1, y_1)(x_2, y_2) = (x_1x_2 + \theta\bar{y}_2y_1, y_2x_1 + y_1\bar{x}_2).$$

Identifying the pairs $(x, 0)$ with the elements x, $\mathfrak{A}$ can be regarded as a subalgebra of $\mathfrak{B}$, and the element $1 = (1, 0)$ is a unity element for $\mathfrak{B}$.

We can easily verify that the mapping

$$(2.3) \qquad w = (x, y) \to \bar{w} = (\bar{x}, -y)$$

* This normed map was first given by Lam in [11; p. 424]. Later, using matrices, he also constructed our normed map $R^{12} \times R^{12} \to R^{26}$ (see (6.7)).

is an involution of $\mathfrak{B}$, and we have

$$t(w) = t(x), \qquad n(w) = n(x) - \theta n(y). \tag{2.4}$$

Define the *commutator* and the *associator*, respectively, by

$$[x, y] = xy - yx, \qquad (x, y, z) = (xy)z - x(yz).$$

Since these expressions are linear in each argument, using the identity $\bar{w} = -w + t(w)1$, it follows that, with a possible change in sign, any bar can be removed. In this form we obtain

$$[x, y] = -[\bar{x}, y] = -[x, \bar{y}] = [\bar{x}, \bar{y}], \tag{2.5}$$

$$(x, y, z) = -(\bar{x}, y, z) = (\bar{x}, \bar{y}, z) = \cdots . \tag{2.6}$$

An algebra $\mathfrak{A}$ over F is *alternative* if $(x, x, y) = 0$, and $(y, x, x) = 0$, for all x, y in $\mathfrak{A}$. In an alternative algebra the associator is skew symmetric in its three variables ([16; p. 27]). Then

$$(x, y, z) = -(y, x, z) = (y, z, x) = \cdots$$

An algebra is *flexible* if

$$(x, y, x) = 0 \tag{2.7}$$

for all x, y in $\mathfrak{A}$. An alternative algebra is flexible, but the converse is not necessarily true. In a flexible algebra we have ([16; p. 28])

$$(x, y, z) + (z, y, x) = 0. \tag{2.8}$$

Let $\theta_1, \cdots, \theta_k$ be k given elements of F, where each $\theta_i \neq 0$. Beginning with $\mathfrak{A}_0 = F$, where $x \to \bar{x} = x$ is the identity transformation on $\mathfrak{A}_0$, by the Cayley-Dickson process we construct $\mathfrak{A}_1 = \mathfrak{A}_0(\theta_1)$, $\mathfrak{A}_2 = \mathfrak{A}_1(\theta_2), \cdots, \mathfrak{A}_k = \mathfrak{A}_{k-1}(\theta_k)$, where each

$$\mathfrak{A}_r = \mathfrak{A}_{r-1}(\theta_r) = F(\theta_1, \cdots, \theta_r) \tag{2.9}$$

is a 2^r-dimensional algebra over F.

$\mathfrak{A}_1$, $\mathfrak{A}_2$ and $\mathfrak{A}_3$ are, respectively, algebras of generalized complex, quaternion and Cayley numbers. As it is known, $\mathfrak{A}_1$ is associative and commutative, $\mathfrak{A}_2$ is associative but is not commutative and $\mathfrak{A}_3$ is alternative but not associative.

Albert has established (see (9.5)) that $\mathfrak{A}_r = \mathfrak{A}_{r-1}(\theta_r)$ is alternative if and only if $\mathfrak{A}_{r-1}$ is associative. Therefore, $\mathfrak{A}_r$ is not alternative for all $t > 3$.

Schafer has proved (see (9.6)) that $\mathfrak{A}_r = \mathfrak{A}_{r-1}(\theta)$ is flexible if and only if $\mathfrak{A}_{r-1}$ is flexible. Consequently, all the algebras $\mathfrak{A}_r$ are flexible.

3. Some properties of the trace and the norm

The definition of the trace trivially implies that $t(x)$ is linear and that $t(x) = t(\bar{x})$. We also have $t(xy) = t(yx)$. This holds since it is equivalent to the identity $[x, y] = [\bar{x}, \bar{y}]$ already stated in (2.5).

The following property, established by Schafer ([15; p. 437]), will be useful If $\mathfrak{A}$ is a flexible algebra, then

$$t(x, y, z) = 0 \tag{3.1}$$

for all x, y, z in $\mathfrak{A}$.

The proof is the following:

$$t(x, y, z) = t((xy)z) - t(x(yz)) = (xy)z + \bar{z}(\bar{y}\bar{x}) - x(yz) - (\bar{z}\bar{y})\bar{x}$$
$$= (x, y, z) - (\bar{z}, \bar{y}, \bar{x}).$$

Now, using first (2.6) and then the flexibility law as stated in (2.8), we get $(\bar{z}, \bar{y}, \bar{x}) = -(z, y, x) = (x, y, z)$, and this ends the proof.

Similarly to the trace, the definition of the norm implies that $n(x)$ is a quadratic function. We have $n(x) = n(\bar{x})$ and $n(\lambda x) = \lambda^2 n(x)$ for $\lambda \in F$.

If $\mathfrak{A}$ is a flexible algebra, then

$$n(xy) = n(x\bar{y}) = n(y\bar{x}) = n(yx), \tag{3.2}$$

for all x, y in $\mathfrak{A}$.

To prove this, we begin with the identity

$$(x\bar{y})(yz) = (xy)(\bar{y}z) - (x, y, z)t(y),$$

that follows trivially from $\bar{y} = -y + t(y)$. Then, substitute $z = \bar{x}$ and use flexibility to obtain $(xy)(\bar{y}\bar{x}) = (x\bar{y})(y\bar{x})$. But this is $n(xy) = n(x\bar{y})$. Now, using $n(w) = n(\bar{w})$, it gives $n(x\bar{y}) = n(y\bar{x})$ and interchanging x and y in the first equality yields $n(y\bar{x}) = n(yx)$. This finishes the proof of (3.2).

Another result we will need is

$$n(z_1 + z_2) = n(z_1) + n(z_2) + t(z_1\bar{z}_2). \tag{3.3}$$

This follows immediately from the expansion of $(z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$.

The next lemma is essential to our construction of norm-preserving maps.

LEMMA (3.4). *Let $\mathfrak{A}_r = \mathfrak{A}_{r-1}(\theta)$ be as in* (2.9) *and suppose that $w_1 = (x_1, y_1)$, $w_2 = (x_2, y_2)$ are given elements in $\mathfrak{A}_r$, where their components x_1, x_2, y_1, y_2 in $\mathfrak{A}_{r-1}$ satisfy the following conditions*:

$$n(x_1x_2) = n(x_1)n(x_2), \qquad n(y_1y_2) = n(y_1)n(y_2),$$
$$n(x_1y_2) = n(x_1)n(y_2), \qquad n(y_1x_2) = n(y_1)n(x_2).$$

Then, we have

$$n(w_1w_2) = n(w_1)n(w_2) + \theta t[x_1(x_2, y_1, y_2)]. \tag{3.5}$$

Proof. Form the product w_1w_2 as in (2.2) and then consider $n(w_1w_2)$ as in (2.4), to get

$$n(w_1w_2) = n(x_1x_2 + \theta\bar{y}_2y_1) - \theta n(y_2x_1 + y_1\bar{x}_2).$$

Then, first use (3.3) to expand the terms and after that remove bars and change the order according to (3.2), so as to have

$$n(w_1w_2) = n(x_1x_2) + \theta^2 n(y_1y_2) - \theta n(x_1y_2) - \theta n(y_1x_2) + \theta \Theta(w_1, w_2)$$

where

$$\Theta(w_1, w_2) = t[(x_1x_2)(\bar{y}_1y_2)] - t[(x_2\bar{y}_1)(y_2x_1)].$$

Now, use the assumptions $n(x_1x_2) = n(x_1)n(x_2)$, etc. in the above expression, to obtain

$$n(w_1w_2) = [n(x_1) - \theta n(y_1)][n(x_2) - \theta n(y_2)] + \theta\Theta(w_1, w_2)$$
$$= n(w_1)n(w_2) + \theta\Theta(w_1, w_2).$$

Finally, we will prove that $\Theta(w_1, w_2)$ is as required in the lemma. From (3.1) and $t(xy) = t(yx)$, we get $t[(x_1x_2)(\bar{y}_1y_2)] = t[x_1(x_2(\bar{y}_1y_2))]$, and $t[(x_2\bar{y}_1)(y_2x_1)] = t[((x_2\bar{y}_1)y_2)x_1] = t[x_1((x_2\bar{y}_1)y_2)]$, therefore,

$$\Theta(w_1, w_2) = -t[x_1(x_2, \bar{y}_1, y_2)] = t[x_1(x_2, y_1, y_2)],$$

and this ends the proof.

We will show that $\Theta(w_1, w_2)$ has several symmetries. Directly from the flexibility law, we have $t[(x_1(x_2, y_1, y_2)] = -t[x_1(y_2, y_1, x_2)]$. Then, from $n(w_1w_2) = n(w_2w_1)$, it follows that $\Theta(w_1, w_2) = \Theta(w_2, w_1)$, hence

$$t[x_1(x_2, y_1, y_2)] = t[x_2(x_1, y_2, y_1)].$$

For simplicity, let $a = x_1$, $b = y_1$, $c = x_2$, $d = y_2$. An alternative use of the above two identities, gives

$$t[a(b, c, d)] = t[b(a, d, c)] = -t[b(c, d, a)] = -t[c(b, a, d)]$$
$$= t[c(d, a, b)] = t[d(c, b, a)] = -t[d(a, b, c)] = -t[a(d, b, c)].$$

4. A normalized basis

Assume that the characteristic of F is not two. As in the known cases of the complex, quaternion and Cayley algebras, Schafer has shown ([15; p. 439]) that a normalized basis may also be chosen for the general $\mathfrak{A}_r$, as follows. Let $m = 2^r$. Then, there is a basis $1, \epsilon_1, \cdots, \epsilon_{m-1}$ for $\mathfrak{A}_r$ such that, for all $1 \leq i,j \leq m - 1$.

$$\epsilon_i^2 = \alpha_i 1 \tag{4.1}$$

$$\epsilon_i\epsilon_j = -\epsilon_j\epsilon_i = \beta_{ij}\epsilon_k, \qquad \text{for } i \neq j, \tag{4.2}$$

where $\alpha_i \neq 0$, $\beta_{ij} \neq 0$ are elements of F depending on $\theta_1, \cdots, \theta_r$, and $\epsilon_k \neq 1$ is an element of the basis uniquely determined by the elements ϵ_i, ϵ_j and different from them.

The existence of the basis is proved easily by induction, as follows. Start with the basis 1, $\epsilon_1 = (0, 1)$ for $\mathfrak{A}_1$, then $\epsilon_1^2 = \alpha_1 1$ where $\alpha_1 = \theta_1 \neq 0$. Now, let a normalized basis $1 = \epsilon_0, \epsilon_1, \cdots, \epsilon_{m-1}$ be given for $\mathfrak{A}_r \subset \mathfrak{A}_{r+1}$. By adding to the elements of this basis, the new elements $\epsilon_i = (0, \epsilon_{i-m})$ for $i = m, m+1, \cdots, 2m-1$, we obtain a normalized basis for $\mathfrak{A}_{r+1}$, as we may readily verify.

From (2.3) and the form $\epsilon_i = (0, \epsilon_k)$ of the elements of the basis, it follows that $\bar{\epsilon}_i = -\epsilon_i$, for all $i \neq 0$. Then, if $x = a_0 1 + \Sigma a_i \epsilon_i$ where $a_k \in F$, we have $\bar{x} = a_0 1 - \Sigma a_i \epsilon_i$, $t(x) = 2a_0$, and using (4.1), (4.2) we get that

$$n(x) = a_0^2 - \Sigma \alpha_i a_i^2. \tag{4.3}$$

Clearly, the elements α_i can be made explicit, if necessary. For example, if $\mathfrak{A}_2 = F(\theta_1, \theta_2)$, for $x \in \mathfrak{A}_2$, we have,

$$n(x) = a_0^2 - \theta_1 a_1^2 - \theta_2 a_2^2 + \theta_2\theta_1 a_3^2, \tag{4.4}$$

and if $\mathfrak{A}_3 = \mathfrak{A}_2(\theta_3)$, for $x \in \mathfrak{A}_3$, we obtain,

$$n(x) = a_0^2 - \theta_1 a_1^2 - \theta_2 a_2^2 + \theta_2\theta_1 a_2^2 - \theta_3[a_4^2 - \theta_1 a_5^2 - \theta_2 a_6^2 + \theta_2\theta_1 a_7^2] \tag{4.5}$$

The quadratic forms (4.4), (4.5) are, respectively, the norms of generalized quaternion and Cayley numbers.

Let $F^m = F \times \cdots \times F$ be the vector space over F constructed as the m-fold product. If to the element $x \in \mathfrak{A}_r$ we assign its ordered components as an element $(a_0, \cdots a_{m-1}) \in F^m$, we set an induced bilinear product

$$F^m \times F^m \to F^m, \tag{4.6}$$

so that $\mathfrak{A}_r$ and F^m become isomorphic as involutorial algebras over F. Under this isomorphism, the normalized basis of $\mathfrak{A}_r$ and the canonical basis of F^m are associated as follows: $\epsilon_0 \leftrightarrow (1, 0, \cdots, 0), \cdots, \epsilon_{m-1} \leftrightarrow (0, \cdots, 0, 1)$. We will make extensive use of this isomorphism.

Finally, Schafer proves (loc. cit.) that, even though $\mathfrak{A}_r$ is not alternative for $r > 3$, the alternative law is fulfilled by all the elements in the normalized basis. That is, $(\epsilon_i, \epsilon_i, \epsilon_j) = (\epsilon_j, \epsilon_i, \epsilon_i) = 0$. This is established using properties (4.1), (4.2) of the basis, and we refer to his paper for a proof.

Here, this result is used as follows. Since the associator is linear in each variable, we have $(\epsilon_i, \epsilon_i, z) = (z, \epsilon_i, \epsilon_i) = 0$ for all $z \in \mathfrak{A}_r$. Let $x = a_0 1 + a_i\epsilon_i$ and $y = b_0 1 + b_j\epsilon_j$. A direct computation gives $(x, y, z) = a_i b_j(\epsilon_i, \epsilon_j, z)$, $(z, x, y) = a_i b_j(z, \epsilon_i, \epsilon_j)$. Consequently, for any z, we have

$$(x, y, z) = (z, x, y) = 0, \qquad \text{if } i = j \tag{4.7}$$

In particular, this holds if x and y are "real" or "complex numbers".

5. Normed maps

Under the isomorphism shown in the preceding section, identify the algebra $\mathfrak{A}_r$ with F^m, the m-fold product of F, where $m = 2^r$. For convenience, an element x of $\mathfrak{A}_r$ will be considered indistinctly as an element x of F^m. Regard F^m as a vector space over F and let $F^p \subset F^m$ be a subspace formed by taking certain p factors of F^m. This is equivalent to specify p elements of the basis $\epsilon_0, \cdots, \epsilon_{m-1}$.

Clearly, the notation F^p is ambiguous but it will be made more precise when necessary.

Assume that the unity $1 = \epsilon_0$ *is always contained* in all the subspaces to be considered. Then, if $x \in F^p$, by (2.1), this assumption implies that $\bar{x} \in F^p$.

Let

$$\phi: F^p \times F^q \to F^s \tag{5.1}$$

be a restriction of the bilinear map (4.6), where F^p, F^q and F^s are three subspaces of F^m. Given $x \in F^p$ and $y \in F^q$ we will show that $xy = \phi(x, y)$ and $yx = \phi(y, x)$, are both in the same subspace F^s. In fact, since $\bar{x} \in F^p$ and $\bar{y} \in F^q$, it follows that $\bar{x}\bar{y} \in F^s$, but $\bar{x}\bar{y} =, \overline{yx}$ therefore, $\overline{yx} \in F^s$ and consequently, $yx \in F^s$.

The bilinear map ϕ is called a *normed map* if

$$n(xy) = n(x)n(y) \tag{5.2}$$

for all $x \in F^p$ and $y \in F^q$. Here, the norm in each subspace is the one obtained by restricting the expression (4.3) to the elements of the normalized basis spanning that subspace.

Let $\mathfrak{A}_r = \mathfrak{A}_{r-1}(\theta_r)$ and $x = (x_1, y_1)$, $y = (x_2, y_2)$ two elements of $\mathfrak{A}_r$ where the components x_1, x_2, y_1, y_2 are in $\mathfrak{A}_{r-1}$ and satisfy the conditions of (3.4). Then, according with (3.5), ϕ is a normed map if and only if

$$\mathcal{O}(x, y) = t[x_1(x_2, y_1, y_2)] = 0. \tag{5.3}$$

If $p = q = s = m$, the condition (5.2) means that $\mathfrak{A}_r$ is a *normed algebra*. For $r \leq 3$ we always have $\mathcal{O}(x, y) = 0$, since $(x_2, y_1, y_2) = 0$. If $r > 3$, $\mathfrak{A}_{r-1}$ is not associative, hence we can find elements x_2, y_1, y_2 in $\mathfrak{A}_{r-1}$ such that $(x_1, y_1, y_2) \neq 0$. By (3.1), the trace of an associator is zero, then, from (4.1), (4.2) it follows that, for some $i > 0$, there exists $x_1 = \epsilon_i$ such that $\mathcal{O}(x, y) \neq 0$. Consequently, the algebra $\mathfrak{A}_r$ constructed by the Cayley-Dickson process is a normed algebra if and only if $r = 0, 1, 2, 3$. In particular, for $r = 2, 3$, we get the normed maps $F^4 \times F^4 \to F^4$ and $F^8 \times F^8 \to F^8$, where the norm is given, respectively, by (4.4) and (4.5).

Now, to study normed maps, we establish the following two lemmas.

LEMMA (5.4). *Given a normed map* $\phi: F^p \times F^q \to F^s$, *restriction of* $F^m \times F^m \to F^m$, *we can always construct the following normed maps*:

$$\phi_1: F^{p+1} \times F^{2q} \to F^{2s}, \tag{5.5}$$

$$\phi_2: F^{2p} \times F^{q+1} \to F^{2s}, \tag{5.6}$$

where both are restrictions of $F^{2m} \times F^{2m} \to F^{2m}$.

Proof. Let $w_1 = (x_1, f)$, $w_2 = (x_2, y_2)$ with $x_1 \in F^p$, $x_2, y_2 \in F^q$ and $f \in F$ where $F = \mathfrak{A}_0$. Consider the product

$$w_1 w_2 = (x_1, f)(x_2, y_2) = (x_1 x_2 + \theta \bar{y}_2 f, y_2 x_1 + f\bar{x}_2)$$

and since $f \in F^p \cap F^q$, we have that each of the two components of w_1w_2 belongs to F^s, then $w_1w_2 \in F^{2s}$. Also the hypotheses of (3.4) are fulfilled and $\Theta(w_1, w_2) = 0$, since $(x_2, f, y_2) = 0$. Therefore, the product w_1w_2 defines the normed map (5.5). Similarly, the map (5.6) is established.

Let ϵ_i be a fixed element of the normalized basis, where $0 < i < m$. Consider $u = f_1\epsilon_0 + f_2\epsilon_i$ and $v = f_3\epsilon_0 + f_4\epsilon_i$, where the f_k's, for $k = 1, 2, 3, 4$, are arbitrary elements of F. If $\epsilon_i = \epsilon_1$, then u, v are "complex numbers".

Define the set $Q(F^p, F^q, \epsilon_i) \subset F^m$ by

$$Q(F^p, F^q, \epsilon_i) = \{vx + uy \mid x \in F^p, y \in F^q, \text{ all } u, v\},$$

and let F^λ be the smallest subspace of F^m containing $Q(F^p, F^q, \epsilon_i)$. It can easily be shown that the number $\lambda = \lambda(F^p, F^q, \epsilon_i)$ depends not only of p and q, but also of the particular subspaces F^p and F^q under consideration.

LEMMA (5.7). *Let $\phi: F^p \times F^q \to F^s$ be a normed map obtained as a restriction of $F^m \times F^m \to F^m$. Assume that $\epsilon_i \in F^p \cap F^q$. Then, there is a normed map*

$$\phi_3: F^{p+2} \times F^{q+2} \to F^{s+\lambda}, \tag{5.8}$$

obtained as a restriction of $F^{2m} \times F^{2m} \to F^{2m}$, where $\lambda = \lambda(F^p, F^q, \epsilon_i)$ is the number defined above.

Proof. Take $w_1 = (x_1, u)$ and $w_2 = (x_2, v)$, where $x_1 \in F^p$, $x_2 \in F^q$ and u, v are as before, and form the product

$$w_1w_2 = (x_1x_2 + \theta\bar{v}u, vx_1 + u\bar{x}_2).$$

Since $\epsilon_0, \epsilon_i \in F^p \cap F^q$, it follows that $\bar{v}u \in F^s$, hence, the first component of w_1w_2 belongs to F^s, and by definition the second component of w_1w_2 is in F^λ, therefore $w_1w_2 \in F^{s+\lambda}$. The assumptions made in the lemma assure that the hypotheses of (3.4) are fulfilled. Also, $\Theta(w_1, w_2) = 0$, since, by (4.7), we have $(x_2, u, v) = 0$. Consequently, the product w_1w_2 defines the map (5.8).

From these lemmas we get the following

COROLLARY (5.9). *Let $\phi: F^p \times F^q \to F^s$ be a normed map obtained as a restriction of $F^m \times F^m \to F^m$. Then, there is a normed map*

$$F^{2p+2} \times F^{2q+3} \to F^{4s+2p+2} \tag{5.10}$$

Proof. Starting with ϕ, from (5.5) and (5.6), we first get $F^{p+1} \times F^{2q} \to F^{2s}$ and then $F^{2p+2} \times F^{2q+1} \to F^{4s}$, where this last map is a restriction of $F^{4m} \times F^{4m} \to F^{4m}$. If $x \in F^{2p+2}$ and $y \in F^{2q+1}$, we can represent these elements by

$$x = ((x_1, f_1), (x_2, f_2)), \qquad y = ((y_1, y_2), (f_3, 0)),$$

where $x_1, x_2 \in F^p$, $y_1, y_2 \in F^q$ and $f_1, f_2, f_3 \in F$. Now, use (5.7) with $u = 0$ and $v = f_4\epsilon_0 + f_5\epsilon_{2m}$ where, $\epsilon_0 = ((1, 0), (0, 0))$ and $\epsilon_{2m} = ((0, 0), (1, 0))$. Then, $(x, 0)(y, v) = (xy, vx)$, and a direct computation of

$$vx = ((f_4, 0), (f_5, 0))((x_1, f_1), (x_2, f_2))$$

shows that $vx \in F^{2p+2}$. Since, $xy \in F^{4s}$, the product $(x, 0)(y, v)$ defines the map (5.10), and this ends the proof.

6. Construction of some normed maps

In order to have a simple and uniform statement about the norm used in every subspace F^p of F^m, from now on, we make the following assumption. *The elements $\theta_1, \cdots, \theta_r$ of F employed to construct the $\mathfrak{A}_r$'s are all equal to* -1. Hence, by induction, it follows easily that $\epsilon_i^2 = -1$ and that $\alpha_i = -1$, for $i = 1, \cdots, m - 1$ (see (4.1)). Consequently, if $x = (x_1, \cdots, x_m)$ is an element of F^m, from (4.3), we have that

$$n(x) = x_1^2 + \cdots + x_m^2 \tag{6.1}$$

is the *usual* norm. Then, a normed map is a bilinear map fulfilling, for the usual norm, the norm condition (5.2). All our maps will be obtained as restrictions of $\mathfrak{A}_r \times \mathfrak{A}_r \to \mathfrak{A}_r$, for a suitable r.

THEOREM (6.2). *We have the following normed maps, gotten as restrictions of $\mathfrak{A}_r \times \mathfrak{A}_r \to \mathfrak{A}_r$, for $r = 4, 5$.*

(6.3) $F^9 \times F^{16} \to F^{16}$, (6.8) $F^{11} \times F^{20} \to F^{32}$,

(6.4) $F^{10} \times F^{10} \to F^{16}$, (6.9) $F^{13} \times F^{13} \to F^{28}$,

(6.5) $F^{10} \times F^{32} \to F^{32}$, (6.10) $F^{11} \times F^{14} \to F^{28}$,

(6.6) $F^{18} \times F^{17} \to F^{32}$, (6.11) $F^{10} \times F^{15} \to F^{31}$,

(6.7) $F^{12} \times F^{12} \to F^{26}$, (6.12) $F^{12} \times F^{14} \to F^{31}$.

Proof: Start with the Cayley multiplication $F^8 \times F^8 \to F^8$ as a normed map. Using the construction (5.5), we immediately get (6.3). Again, applying (5.8), with $\epsilon_i = \epsilon_1$, we have $\lambda = 8$, and (6.4) follows.

Now, with (6.3) already as a normed map, and applying to it the constructions (5.5) and (5.6) we, respectively, obtain (6.5) and (6.6).

Again, applying the construction (5.8), with $\epsilon_i = \epsilon_1$, to the map (6.4), it can be easily verified that we get $\lambda = 10$, and hence, (6.7) follows.

The map (6.8) is obtained by using the construction (5.5) in (6.4).

The map (6.9) is a restriction of (6.6). We will show directly how it is constructed. Let $w_1 = ((x_1, q_1), (f_1, 0))$ and $w_2 = ((x_2, f_2), (q_2, 0))$ be two elements, where $f_1, f_2 \in \mathfrak{A}_0 = F$, $q_1, q_2 \in \mathfrak{A}_2$ and $x_1, x_2 \in \mathfrak{A}_3$. The w_1 and w_2 represent, respectively, the form of the elements in the left and in the right factor of $F^{13} \times F^{13}$. Set $y_1 = (x_1, q_1)$, $y_2 = (x_2, f_2)$, $z_1 = (f_1, 0)$ and $z_2 = (q_2, 0)$. Then, $w_1 = (y_1, z_1)$ and $w_2 = (y_2, z_2)$. Obviously, we have all the conditions of (3.4) fulfilled and also $\mathcal{O}(w_1, w_2) = 0$. The product w_1w_2 defines the map and we only need to prove that $w_1w_2 \in F^{28}$. We have

$$w_1w_2 = (y_1, z_1)(y_2, z_2) = (y_1y_2 - \bar{z}_2z_1, z_2y_1 + z_1\bar{y}_2),$$

and $y_1y_2 - \bar{z}_2z_1 \in F^{16}$. Now, for the second component we have $z_2y_1 + z_1\bar{y}_2 =$

$(q_2x_1 + f_1\bar{x}_2, q_1q_2 - f_2f_1)$, therefore, $z_2y_1 + z_1\bar{y}_2 \in F^{12}$, and this ends the proof of (6.9).

The map (6.10) is constructed as follows. Set, $w_1 = ((x_1, u_1), (f, 0))$ and $w_2 = ((x_2, u_2), (q, 0))$, where $f \in F$, $u_1, u_2 \in \mathfrak{A}_1$, $q \in \mathfrak{A}_2$ and $x_1, x_2 \in \mathfrak{A}_3$. Clearly, $w_1 \in F^{11}$ and $w_2 \in F^{14}$. If $y_1 = (x_1, u_1)$, $y_2 = (x_2, u_2)$, $z_1 = (f, 0)$ and $z_2 = (q, 0)$, then $w_1 = (y_1, z_1)$ and $w_2 = (y_2, z_2)$. Again, we have all the conditions of (3.4) fulfilled and $\mathcal{O}(w_1, w_2) = 0$. As before, the product w_1w_2 defines the map and we only need to verify that $w_1w_2 \in F^{28}$. This follows similarly to the preceding case and the proof is omitted.

To construct (6.11), represent the elements $w_1 \in F^{10}$ and $w_2 \in F^{15}$, as follows: $w_1 = ((x_1, u_1), (f_1, 0))$ and $w_2 = ((x_2, u_2), (q, f_2))$, where $f_1, f_2 \in \mathfrak{A}_0$, u_1, $u_2 \in \mathfrak{A}_1$, $q \in \mathfrak{A}_2$ and $x_1, x_2 \in \mathfrak{A}_3$, with the last component of x_1 (*i.e.*, the coefficient of ϵ_7) *always* equal to zero.

Set $y_1 = (x_1, y_1)$, $y_2 = (x_2, u_2)$, $z_1 = (f_1, 0)$ and $z_2 = (q, f_2)$. Then, $w_1 = (y_1, z_1)$ and $w_2 = (y_2, z_2)$. The conditions of (3.4) are fulfilled and $\mathcal{O}(w_1, w_2) = 0$, hence, the product w_1w_2 defines a normed map.

To show that $w_1w_2 \in F^{31}$, as before, consider the two components of $w_1w_2 = (y_1, z_1)(y_2, z_2)$. The first component $y_1y_2 - \bar{z}_2z_1 \in F^{16}$. From the second component, with a further expansion, we get

$$z_2y_1 + z_1\bar{y}_2 = (qx_1 - u_1f_2 + f_1\bar{x}_2, u_1q + f_2\bar{x}_1 - u_2f_1),$$

where $u_1q + f_2\bar{x}_1 - u_2f_1 \in F^7$, since the last component of $f_2\bar{x}_1$ is zero. Therefore, $z_2y_1 + z_1\bar{y}_2 \in F^{15}$, and this ends the proof of (6.11).

For (6.12), represent the elements $w_1 \in F^{10}$ and $w_2 \in F^{14}$, as follows: $w_1 = ((x_1, q_1), (f_1, 0))$ and $w_2 = ((x_2, f_2), (q_2, f_3))$, where $f_1, f_2 \in \mathfrak{A}_0$, $q_1, q_2 \in \mathfrak{A}_2$ and $x_1, x_2 \in \mathfrak{A}_3$, with the last component of x_1 equal to zero. We skip the rest of the proof, since it is similar to the one given for (6.11).

THEOREM (6.13). *We obtain the following normed maps, as restrictions of* $\mathfrak{A}_r \times \mathfrak{A}_r \to \mathfrak{A}_r$, *for* $r = 6, 7$.

(6.14) $F^{11} \times F^{64} \to F^{64}$, (6.22) $F^{20} \times F^{19} \to F^{58}$,

(6.15) $F^{20} \times F^{33} \to F^{64}$, (6.23) $F^{19} \times F^{19} \to F^{56}$,

(6.16) $F^{19} \times F^{34} \to F^{64}$, (6.24) $F^{22} \times F^{23} \to F^{86}$,

(6.17) $F^{36} \times F^{18} \to F^{64}$, (6.25) $F^{24} \times F^{23} \to F^{96}$,

(6.18) $F^{22} \times F^{21} \to F^{64}$, (6.26) $F^{23} \times F^{23} \to F^{94}$,

(6.19) $F^{13} \times F^{24} \to F^{52}$, (6.27) $F^{26} \times F^{25} \to F^{104}$,

(6.20) $F^{14} \times F^{26} \to F^{56}$, (6.28) $F^{28} \times F^{27} \to F^{112}$,

(6.21) $F^{18} \times F^{19} \to F^{50}$, (6.29) $F^{37} \times F^{36} \to F^{128}$.

Proof. Applying the construction (5.5), respectively, to (6.4), (6.6), (6.7) and (6.9), we get (6.14), (6.16), (6.19) and (6.20). In the same way, using the construction (5.6), respectively, with (6.5), (6.6) and (6.8), we obtain (6.15), (6.17) and (6.18). Again, from (5.5) and (6.17) we have (6.29), and with (5.6) on (6.19) and (6.20), we get (6.27) and (6.28).

The construction (5.10) applied to $F^8 \times F^8 \to F^8$ and to (6.4), gives, respectively, the maps (6.21) and (6.24).

To establish (6.22) we use (5.8), with $\epsilon_i = \epsilon_1$, on the map $F^{18} \times F^{17} \to F^{32}$, given by (6.6). If $w_1 \in F^{18}$ and $w_2 \in F^{17}$, we have $w_1 = ((x_1, f_1), (x_2, f_2))$ and $w_2 = ((x_3, x_4), (f_3, 0))$, where $f_1, f_2, f_3 \in \mathfrak{A}_0$ and $x_1, x_2, x_3, x_4 \in \mathfrak{A}_3$. Then, if $u, v \in \mathfrak{A}_1$, we have that $vw_1 + uw_2 \in F^{26}$, consequently, $\lambda = 26$ and (6.22) follows.

To prove (6.23), restrict (6.6) to $F^{17} \times F^{17} \to F^{32}$, by taking $w_1 = ((x_1, f_1), (x_2, 0))$ and w_2 as above.

Now, using the construction (5.8) on this restriction, we get that $vw_1 + uw_2 \in F^{24}$, therefore $\lambda = 24$ and (6.23) follows.

As before, using (5.8), with $\epsilon_i = \epsilon_1$, on (6.18) and on its restriction $F^{21} \times F^{21} \to F^{64}$, the cases (6.25) and (6.26) are proved in a similar form. We omit the details and this ends the proof of (6.13).

The following is a table of the first bilinear normed maps of the form $F^p \times F^p \to F^s$, known to exist.

(6.30) *Normed maps of the form* $F^p \times F^p \to F^s$

p	s	p	s	p	s	p	s
1	1	9	16	17	32	25	104
2	2	10	16	18	50	26	112
3	4	11	26	19	56	27	112
4	4	12	26	20	64	28	128
5	8	13	28	21	64	·	·
6	8	14	32	22	86	·	·
7	8	15	32	23	94	·	·
8	8	16	32	24	104	36	128

The first eight maps of this table are constructed using the multiplication of $\mathfrak{A}_r$ for $r \leq 3$. The other maps are obtained from (6.2) and (6.13).

7. Some general formulas

We will establish certain general formulas about families of normed maps. The construction is by induction, as follows. Let $a = (a_1, \cdots, a_r, \cdots)$, $b = (b_1, \cdots, b_r, \cdots)$, $c = (c_1, \cdots, c_r, \cdots)$, be three given sequences, where each of the elements a_r, b_r, c_r is either zero or one. Consider the constructions (5.5), (5.6) and (5.8) where, for simplicity, the last one is always taken with $\epsilon_i = \epsilon_1$. Starting with the tern (a_1, b_1, c_1) and with the normed map $\phi: F^8 \times F^8 \to F^8$, given by the multiplication of $\mathfrak{A}_3$, first, apply to ϕ the construction (5.5) if $a_1 = 1$, or "the identity construction" (to leave it as it is) if $a_1 = 0$. In this form, we get the map ϕ' and $\phi' = \phi$ if $a_1 = 0$. Then, using b_1 and (5.6), do the same to ϕ' to have the map ϕ''. Finally, with c_1 and (5.8), and using this method on ϕ'', we obtain the map ϕ_1. We will regard ϕ_1 as determined from ϕ by the tern (a_1, b_1, c_1), and this begins the induction.

Now, suppose given ϕ_{r-1}. The r^{th} step of the induction is accomplished applying the above procedure to ϕ_{r-1} and the tern (a_r, b_r, c_r), to get the map ϕ_r.

Before writing a formula for ϕ_r, we will work an explicit expression for ϕ_1. In order to avoid the indexes, set $a_1 = \alpha$, $b_1 = \beta$ and $c_1 = \gamma$.

THEOREM (7.1). *The ϕ_1 constructed above produces the following normed map,*

$$\phi_1 : F^{2^\beta(8+\alpha)+2\gamma} \times F^{2^\alpha 8+\beta+2\gamma} \to F^{2^{\alpha+\beta+\gamma}8-6\alpha\beta\gamma}$$

obtained as a restriction of $\mathfrak{A}_s \times \mathfrak{A}_s \to \mathfrak{A}_s$, for $s = \alpha + \beta + \gamma + 3$. We recall that each α, β, γ is either zero or one.

Proof. As above, using (5.5) combined with α, on the map ϕ, we get

$$\phi' : F^{8+\alpha} \times F^{8+8\alpha} \to F^{8+8\alpha}.$$

Since $\alpha = 0, 1$, we have that

$$2^\alpha = 1 + \alpha. \tag{7.2}$$

Hence, $\phi' : F^{8+\alpha} \times F^{2^\alpha 8} \to F^{2^\alpha 8}$. If $w_1 \in F^{8+\alpha}$ and $w_2 \in F^{2^\alpha 8}$, then $w_1 = (x, \alpha f)$ and $w_2 = (y_1, \alpha y_2)$, where $x_1, y_1, y_2 \in F^8$ and $f \in \mathfrak{A}_0$. If $\alpha = 0$, then we identify x with $(x, 0)$ and y_1 with $(y_1, 0)$. We have, $\phi'(w_1, w_2) = w_1 w_2$ and $w_1 w_2 \in F^{2^\alpha 8}$.

Now, using (5.6) combined with β, on ϕ', we obtain

$$\phi'' : F^{8+\alpha+(8+\alpha)\beta} \times F^{2^\alpha 8+\beta} \to F^{2^\alpha 8+2^\alpha 8\beta},$$

and, after simplifying this expression with the identity (7.2), we have

$$\phi'' : F^{2^\beta(8+\alpha)} \times F^{2^\alpha 8+\beta} \to F^{2^{\alpha+\beta}8}.$$

If $w_1' \in F^{2^\beta(8+\alpha)}$ and $w_2' \in F^{2^\alpha 8+\beta}$, then

$$w_1' = ((x_1, \alpha f_1), \beta(x_2, \alpha f_2)), \tag{7.3}$$

$$w_2' = ((y_1, \alpha y_2), \beta(f_3, 0)), \tag{7.4}$$

where $x_1, x_2, y_1, y_2 \in F^8$ and $f_1, f_2, f_3 \in \mathfrak{A}_0$. Then, $\phi''(w_1', w_2') = w_1' w_2'$ and $w_1' w_2' \in F^{2^{\alpha+\beta}8}$.

Finally, using (5.8) combined with γ, on ϕ'', we get

$$\phi_1 : F^{2^\beta(8+\alpha)+2\gamma} \times F^{2^\alpha 8+\beta+2\gamma} \to F^{2^{\alpha+\beta}8+\lambda},$$

where the number $\lambda = \lambda(F^{2^\beta(8+\alpha)}, F^{2^\alpha 8+\beta}, \epsilon_1)$ is as in (5.7), and needs to be computed.

If $w_1'' \in F^{2^\beta(8+\alpha)+2\gamma}$ and $w_2'' \in F^{2^\alpha 8+\beta+2\gamma}$, then

$$w_1'' = [((x_1, \alpha f_1), \beta(x_2, \alpha f_2)), \gamma((z_1, 0), (0, 0))], \tag{7.5}$$

$$w_2'' = [((y_1, \alpha y_2), \beta(f_3, 0)), \gamma((z_2, 0), (0, 0))], \tag{7.6}$$

where the first two parts of w_1'' and w_2'' are as in (7.3) and (7.4), and $z_1, z_2 \in \mathfrak{A}_1$. With the correct inclusions, we can write $w_1'' = (w_1', \gamma z_1)$, $w_2'' = (w_2', \gamma z_2)$, and $\phi_1(w_1'', w_2'') = w_1''w_2''$. Now,

$$w_1''w_2'' = (w_1'w_2' - \gamma\bar{z}_2z_1, \gamma z_2w_1' + \gamma z_1\bar{w}_2'),$$

with $w_1'w_2' - \gamma\bar{z}_2z_1 \in F^{2^{\alpha+\beta}8}$ and $\gamma(z_2w_1' + z_1\bar{w}_2') \in F^{\lambda}$, where λ is the dimension of the smallest subspace containing the general term $v = \gamma(z_2w_1' + z_1\bar{w}_2')$. A direct computation gives

$$v = \gamma[(z_2x_1 + z_1\bar{y}_1, \alpha(f_1z_2 - y_2z_1)), (\beta(x_2z_2 - f_3z_1), \alpha\beta f_2\bar{z}_2)],$$

and an inspection of the above expression, shows that $\lambda = \gamma(8 + 8\alpha + 8\beta + 2\alpha\beta)$ Then, using identities of the type (7.2), it follows that

$$\lambda = 8\gamma(1 + \alpha)(1 + \beta) - 6\alpha\beta\gamma = 8\gamma 2^{\alpha+\beta} - 6\alpha\beta\gamma.$$

Hence, $2^{\alpha+\beta}8 + \lambda = 2^{\alpha+\beta+\gamma}8 - 6\alpha\beta\gamma$, as stated in the theorem.

From (7.5), (7.6), it follows that the total dimension of the vector space used is not greater than 64. In terms of α, β, γ, this dimension is equal to $8(1 + \alpha + \beta + \alpha\beta)(1 + \gamma) = 2^{\alpha+\beta+\gamma+3}$. Therefore, the normed map ϕ_1 is a restriction as pointed in the theorem, and this ends the proof of (7.1).

The following remarks can be made. If we consider all the possible values of α, β, γ, the formula in (7.1) covers eight cases. Of these, only four are relevant and have been already established. They are (6.3), (6.4), (6.6) and (6.22). Some of the other cases, as (6.7) and (6.8), will be obtained with the next map ϕ_2.

Now, we will consider the general case. As before, let a, b, c be three arbitrary sequences formed by zeros and ones. For every nonnegative integer r we will define the functions $\Phi_r(a, b, c)$ and $\Psi_r(a, b, c)$ by induction, as follows. First, $\Phi_0(a, b, c) = \Psi_0(a, b, c) = 8$.
Then,

$$\Phi_r(a, b, c) = 2^{b_r}[\Phi_{r-1}(a, b, c) + a_r] + 2c_r \tag{7.7}$$

$$\Psi_r(a, b, c) = 2^{a_r}\Psi_{r-1}(a, b, c) + b_r + 2c_r. \tag{7.8}$$

Let $\Phi_r(2a, b, c)$ and $\Psi_r(a, 2b, c)$ represent the values obtained by formal substitution in (7.7) and (7.8) of the sequences $(2a, b, c)$ and $(a, 2b, c)$, where $2a = (2a_1, 2a_2, \cdots)$ and $2b = (2b_1, 2b_2, \cdots)$. Again, by induction, we define the function $\Lambda_r(a, b, c)$, as follows: $\Lambda_0(a, b, c) = 8$ and, for $r \geq 1$,

$$\begin{aligned}\Lambda_r(a, b, c) = {} & 2^{a_r+b_r}\Lambda_{r-1} + c_r[2^{b_r}\Phi_{r-1}(2a, b, c) + 2^{a_r}\Psi_{r-1}(a, 2b, c) \\ & + 2a_rb_r - 8 - 2\textstyle\sum_{i=1}^{r-1} (a_i + b_i + c_i)].\end{aligned} \tag{7.9}$$

As a generalization of (7.1), we have the following

THEOREM (7.10). *There is a normed map* $\phi_r: F^{\Phi_r(a,b,c)} \times F^{\Psi_r(a,b,c)} \to F^{\Lambda_r(a,b,c)}$ *obtained as a restriction of* $\mathfrak{A}_s \times \mathfrak{A}_s \to \mathfrak{A}_s$, *where* $s = 3 + \sum_{i=1}^{r} (a_i + b_i + c_i)$.

The proof is by induction and requires analogous work to the one developed for (7.1). The long expressions, like (7.9), make the proof somehow involved, and for this reason it is omitted.

Many of the normed maps of (6.2), (6.13) are obtained from (7.10), for $r = 1, 2$, with suitable values of a, b, c. However, when r increases, the Λ_r becomes "big" and, in general, (7.10) may not be efficient enough for constructing normed maps. Perhaps this is so because we consider only normed maps that are restrictions of the multiplication of an algebra.

8. Discussion of some problems

Let $\phi: F^p \times F^q \to F^s$ be a bilinear map and set $xy = \phi(x, y)$. The map is said to be *nonsingular* if $xy = 0$ implies $x = 0$ or $y = 0$.

If F is a *formally real* field and if the norm is as (6.1), then, it follows that every normed map can be regarded as a nonsingular bilinear map. In fact, if $xy = 0$, then $n(xy) = n(x)n(y) = 0$, hence, $x = 0$ or $y = 0$.

The following problem was considered by Hurwitz in [8; p. 570]:

(8.1) *Given the integers p and q, determine the smallest integer* $k = \alpha(p, q, F)$ *for which there is a normed map* $F^p \times F^q \to F^k$.

On the other hand, related to some problems in algebraic topology, the following similar question has been considered (cf. [2], [12], [13]).

(8.2) *Given the integers p and q, determine the smallest integer* $s = \beta(p, q, F)$ *for which there is a nonsingular bilinear map* $F^p \times F^q \to F^s$.

We write $\alpha(p, q) = \alpha(p, q, R)$ and $\beta(p, q) = \beta(p, q, R)$, for brevity, if $F = R$ is the real field. Clearly, if F is formally real, from the above remark, it follows that $\alpha(p, q, F) \geq \beta(p, q, F)$. So, for the real field, we have

$$(8.3) \qquad \alpha(p, q) \geq \beta(p, q).$$

Usually, to compute $\beta(p, q)$, we first construct a nonsingular bilinear map $R^p \times R^q \to R^s$ and then prove that $s - 1 < \beta(p, q)$. Regarding this second part, the following theorem of Hopf gives a necessary condition for the number $\beta(p, q)$.

THEOREM (8.4). *If* $s \geq \beta(p, q)$, *then the binomial coefficients* $\binom{s}{i}$ *are even numbers for all* $s - p < i < q$.

The original proof of Hopf was one of the earliest applications of the cohomology ring of a space ([7], [10]). Later, an algebraic proof was given by Behrend ([4]), using the theory of intersections for algebraic varieties, and extending the result for *real closed* fields.

Now, we will indicate some special cases where $\alpha(p, q)$ has been determined. As defined in [1; p. 625], let $\phi(n)$ be the number of integers t with $0 < t \leq n$, and $t \equiv 0, 1, 2$ or $4 \pmod 8$. Then, if $q = 2^{\phi(p)}$, from the known facts about the Hurwitz-Radon maps ([6], [8], [14]), it follows that $\alpha(p + 1, q) = q$. In particular, if $p = 8$, then $\phi(8) = 4$, and we have $R^9 \times R^{16} \to R^{16}$ where $\alpha(9, 16) = 16$ (cf. (6.3)).

For $p = q$, we have that, $\alpha(1, 1) = 1$, $\alpha(2, 2) = 2$, $\alpha(3, 3) = \alpha(4, 4) = 4$, $\alpha(5, 5) = \alpha(6, 6) = \alpha(7, 7) = \alpha(8, 8) = 8$ and $\alpha(9, 9) = 16$. The maps of the assertions come from the classical normed algebras, excepting the last one that is obtained as a restriction of $R^9 \times R^{16} \to R^{16}$ (cf. table (6.30)). In these cases, (8.4) can be used to verify that they occur in the smallest possible dimension. For example, we cannot have $15 = \alpha(9, 9) \geq \beta(9, 9)$, since in (8.4), the first binomial coefficient $\binom{15}{7}$ is an odd number.

These normed maps were known since 1922. It seems that the first new normed map $R^{10} \times R^{10} \to R^{16}$ was given by Lam in [11; p. 424] (see footnote at the introduction) and, essentially, it is like our map (6.4). Clearly, $\alpha(p, p) \leq \alpha(p + 1, p + 1)$, therefore, $\alpha(10, 10) = 16$. If F is any field of characteristic different from two, from (6.4) it follows that $\alpha(9, 9, F) \leq \alpha(10, 10, F) \leq 16$. In general, in this case we cannot apply (8.4), and we have the following

Problem (8.5). *Is* $\alpha(9, 9, F) = \alpha(10, 10, F) = 16$?

The same question could be asked for other normed maps. For example: *Is* $\alpha(5, 5, F) = 8$? I do not know the answer, even if F is the Galois field $GF(3)$.

Problem (8.6). *Find the values of* $\alpha(11, 11)$, $\alpha(12, 12)$ *and* $\alpha(13, 13)$.

It was established by Lam (*loc. cit.*) that $\beta(11, 11) = \beta(12, 12) = 17$ and $\beta(13, 13) = 19$. We like to remark that the second part of the proof of these cases cannot be handled with (8.4). With base on (6.7) and (6.9), *we conjecture that* $\alpha(11, 11) = \alpha(12, 12) = 26$ *and that* $\alpha(13, 13) = 28$.

Finally, from (6.6) and with the help of (8.4), we can prove that $\alpha(18, 17) = \beta(18, 17) = 32$. Here, *we conjecture that* $\alpha(14, 14) = \alpha(15, 15) = \alpha(16, 16) = 32$, where the maps are gotten as restrictions of the normed map (6.6).

Appendix
9. The associator

Let $\mathfrak{A}$ be an algebra, with an involution, over a field F. By the Cayley-Dickson process construct the algebra $\mathfrak{B} = \mathfrak{A}(\theta)$, where $\theta \neq 0$ is a fixed element of F.

Given $x = (a, \alpha)$, $y = (b, \beta)$, $z = (c, \gamma)$, elements of $\mathfrak{B}$, we will write the associator (x, y, z) explicitly, using associators and commutators on the elements $a, b, c, \alpha, \beta, \gamma$ in $\mathfrak{A}$.

Let A, B, C, D be defined as follows:

$$A = (a, b, c) + \theta[(c, \beta, \alpha) - (b, \gamma, \alpha) + (\gamma, \beta, a) - (\gamma, \alpha, b)],$$

$$B = \theta(-[a, \bar{\gamma}\beta] + [b, \bar{\gamma}\alpha] - [c, \bar{\beta}\alpha]),$$

$$C = -\theta(\alpha, \beta, \gamma) + (\alpha, b, c) - (\gamma, b, a) + (\beta, c, a) - (\beta, a, c),$$

$$D = \alpha[b, c] - \beta[a, c] + \gamma[a, b] + \theta(\gamma[\bar{\beta}, \alpha] + [\gamma, \alpha\bar{\beta}]).$$

Then, we have

$$(9.1) \qquad (x, y, z) = (A + B, C + D).$$

To establish this, write $(xy)z - x(yz)$ explicitly in terms of the elements of $\mathfrak{A}$. Then, to have only associators and commutators use seven trivial identities of the following type (we only write three of them, since the other four are similar):

$$(c\bar{\beta})\alpha - (\bar{\beta}\alpha)c = (c, \bar{\beta}, \alpha) + [c, \bar{\beta}\alpha],$$
$$(\bar{b}\bar{\gamma})\alpha - \bar{\gamma}(\alpha\bar{b}) = (\bar{\gamma}, \alpha, \bar{b}) + (\bar{b}, \bar{\gamma}, \alpha) + [\bar{b}, \bar{\gamma}\alpha],$$
$$\gamma(\bar{\beta}\alpha) - \alpha(\bar{\beta}\gamma) = (\alpha, \bar{\beta}, \gamma) + \gamma[\bar{\beta}, \alpha] + [\gamma, \alpha\bar{\beta}].$$

Finally, remove bars from associators and commutators, accordingly with (2.5), (2.6), and (9.1) follows.

Considering particular cases of (9.1), we get

$$(9.2) \qquad (x, x, y) = ((a, a, b) + \theta(b, \alpha, \alpha) - \theta(a, \beta, \alpha); - \theta(\alpha, \alpha, \beta) - (\beta, a, a) + (\alpha, b, a)),$$

$$(9.3) \qquad (y, x, x) = ((b, a, a) + \theta(\alpha, \alpha, b) - \theta(\alpha, \beta, a); - \theta(\beta, \alpha, \alpha) + (\beta, a, a) - (\alpha, b, a)),$$

$$(9.4) \qquad (x, y, x) = ((a, b, a) + \theta[(a, \beta, \alpha) + (\alpha, \beta, a) - (b, \alpha, \alpha) - (\alpha, \alpha, b)]; 0).$$

Using the first two of these expressions we will establish the following result due to Albert ([3]; p. 172).

LEMMA (9.5). *The algebra $\mathfrak{B} = \mathfrak{A}(\theta)$ is alternative if and only if the algebra $\mathfrak{A}$ is associative.*

Proof. Let $\mathfrak{A}$ be associative, then from (9.2), (9.3) we have that $(x, x, y) = 0$ and $(y, x, x) = 0$, therefore $\mathfrak{B}$ is alternative. Now, suppose that $\mathfrak{B}$ is alternative, we have $(x, x, y) = 0$, and from (9.2) with $\beta = 0$, it follows that $(\alpha, b, a) = 0$, so $\mathfrak{A}$ is associative.

In a similar form we will now prove the following lemma due to Schafer ([15; p. 437]).

LEMMA (9.6). *The algebra $\mathfrak{B} = \mathfrak{A}(\theta)$ is flexible if and only if the algebra $\mathfrak{A}$ is flexible.*

Proof. Since $\mathfrak{A} \subset \mathfrak{A}(\theta)$ as a subalgebra, one part of the statement follows. For the other part, if $\mathfrak{A}$ is flexible, from (9.4) and (2.8) we have $(x, y, x) = 0$, and this ends the proof.

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REFERENCES

[1] J. F. ADAMS, *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–32.
[2] J. ADEM, *Some immersions associated with bilinear maps*, Bol. Soc. Mat. Mexicana **13** (1968), 95–104.

[3] A. A. ALBERT, *Quadratic forms permitting composition,* Ann. of Math. **43** (1942) 161–77.
[4] F. BEHREND, *Über Systeme reeler algebraischer Gleichungen,* Compositio Math. **7** (1939) 1–19.
[5] L. E. DICKSON, *On quaternions and their generalization and the history of the eight square theorem,* Ann. of Math. **20** (1919), 155–71.
[6] B. ECKMANN, *Gruppentheoretischer Beweis von Hurwitz-Radon über die Komposition quadratischer Formen,* Comment. Math. Helv. **15** (1942) 358–66.
[7] H. HOPF, *Ein topologischer Beitrag zur reelen Algebra,* Comment, Math. Helv. **13** (1940–41) 219–39.
[8] A. HURWITZ, *Über die Komposition der quadratische Formen von beliebig vielen Variabeln,* Nachr. K. Gesellesch. Göttingen M. Ph. Kl (1898), 309–16.
[9] A. HURWITZ, *Über die Komposition der quadratischen Formen,* Math. Ann. **88** (1923), 1–25.
[10] I. M. JAMES, *Two problems studied by Heinz Hopf,* in Lecture Notes in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, **279** (1972), 134–60.
[11] K. Y. LAM, *Construction of nonsingular bilinear maps,* Topology **6** (1967), 423–26.
[12] K. Y. LAM, *Construction of some nonsingular bilinear maps,* Bol. Soc. Mat. Mexicana **13** (1968), 88–94.
[13] R. J. MILGRAM, *Immersing projective spaces,* Ann. of Math. **85** (1967), 473–82.
[14] J. RADON, *Lineare Schären orthogonalen Matrizen,* Abh. Math. Sem. Univ. Hamburg, **1** (1922), 1–14.
[15] R. D. SCHAFER, *On the algebras formed by the Cayley-Dickson process,* Amer. J. Math. **76** (1954), 435–46.
[16] R. D. SCHAFER, An introduction to nonassociative algebras, Academic Press, New York-London, 1966.