THE COHOMOLOGY OF THE SPECTRUM bJ

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The spectrum bJ has been very useful in solving several classical questions in homotopy theory [5], [7]. Its homotopy groups follow immediately from [1] and [3]; in this paper we compute the $\mathcal{A}$-module $H^*(bJ)$ and $\operatorname{Ext}_{\mathcal{A}}(H^*(bJ), Z_2)$. (All cohomology groups have Z_2 coefficients.)

Let $\mathcal{A}_n$ denote the subalgebra of the Steenrod algebra $\mathcal{A}$ generated by $Sq^1, \cdots, Sq^{2^n}$. $\operatorname{Ext}_{\mathcal{A}_2}(Z_2, Z_2)$ has been computed in [6] to be a bigraded algebra over Z_2 with 13 generators and 54 relations. Among the generators are elements h_0, h_1, ω of bidegree $(s, t) = (1, 1)$, $(1, 2)$ and $(4, 12)$, respectively. If M is a graded $\mathcal{A}_2$-module, we picture $\operatorname{Ext}_{\mathcal{A}_2}^{s,t}(M, Z_2)$ on a graph with horizontal coordinate $t - s$ and vertical coordinate s, letting vertical lines denote Yoneda multiplication by h_0 and diagonal lines denote multiplication by h_1, and similarly for $\mathcal{A}$-modules. A "tower" is a subset of $\operatorname{Ext}^{s,t}(M, Z_2)$ consisting of elements $x, h_0x, h_0^2x, \cdots$ for some x.

Then $\operatorname{Ext}_{\mathcal{A}_2}^{s,t}(Z_2, Z_2)$ begins as in Table 1.

Our main result is

Theorem 1. i) *$H^*(bJ)$ is the $\mathcal{A}$-module with generators g_0 and g_7 (of degree 0 and 7, respectively) and relations Sq^1g_0, Sq^2g_0, Sq^4g_0, $Sq^8g_0 + Sq^1g_7$, $S^{7s}g_7$*, and $(Sq^4Sq^6 + Sq^7Sq^3)g_7$.

ii) $\operatorname{Ext}_{\mathcal{A}}^{s,t}(H^*bJ, Z_2) \approx A^{s,t} \oplus B^{s+2,t+1}$, *where* $A^{s,t} \approx \operatorname{Ext}_{\mathcal{A}_2}^{s,t}(Z_2, Z_2)$ *without the towers $h_0^i\omega^{2j+1}$, $i, j \geq 0$, and $B^{s,t} \approx \operatorname{Ext}_{\mathcal{A}_2}^{s,t}(Z_2, Z_2)$ without $\omega^i x^{s,t}$ for all $x^{s,t}$ such that $t - s \leq 3$, and with infinite towers built upon $\omega^{2i+1}h_2^2$ and towers of height four built upon $\omega^{2i}h_2^2$.*

Thus $\operatorname{Ext}_{\mathcal{A}}^{s,t}(H^*(bJ), Z_2)$ begins as in Table 2. Note that there will be many nonzero differentials in the Adams spectral sequence for $\pi_*(bJ)$. Part (i) implies that $H^*(bJ)$ is a free $\mathcal{A}//\mathcal{A}_3$-module, and hence $\operatorname{Ext}_{\mathcal{A}}(H^*bJ, Z_2) \approx \operatorname{Ext}_{\mathcal{A}_3}\cdot(M, Z_2)$, where M has the generators and relations as in part (i).

As in [8] bo and bsp denote the connected Ω-spectra whose $(8k)$th spaces are $BO(8k, \infty)$ and $BSp(8k, \infty) = \Omega^4BO(8k + 4, \infty)$, respectively. All spaces are localized at 2. (bsp was denoted by bo^4 in [5] and [7]). The Adams operation $\psi^3 - 1$ induces a map $bo \xrightarrow{\theta} \Sigma^4bsp$. bJ is defined to be the fibre of θ. From [1; 5.2, 8.1], the homotopy sequence of θ, and [3; 1.3] we easily see

Proposition 2.

$$\pi_i(bJ) = \begin{cases} 0 & i \equiv 4, 5, 6\ (8) \\ Z_2 & i \equiv 0, 2\ (8)\ (\textit{except } i = 0) \\ Z_2 \oplus Z_2 & i \equiv 1\ (\textit{except } i = 1) \\ Z_2j & i + 1 = 2^{j-1}.\ \textit{odd}\ (j \geq 3). \end{cases}$$

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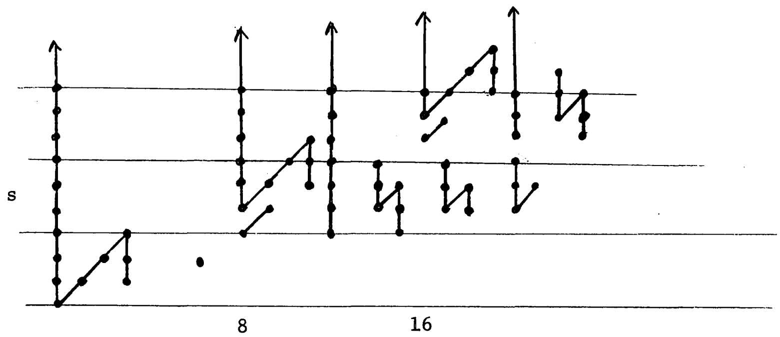
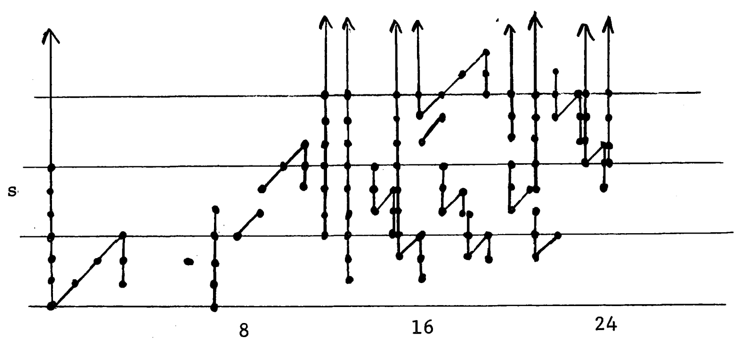
$t-s$

Table 1: $\mathrm{Ext}_{\mathfrak{a}_2}^{s,t}(Z_2, Z_2)$



$t-s$

Table 2: $\mathrm{Ext}_{\mathfrak{a}}^{s,t}(H^*(bJ), Z_2)$



Proof of Theorem 1. $H^*(bo)$ and $H^*(bsp)$ are well-known [10] to be $\mathfrak{a}//\mathfrak{a}_1$ and $\mathfrak{a}/\mathfrak{a}(Sq^1, Sq^5)$, respectively. $\mathrm{Ext}_{\mathfrak{a}}(H^*(bo), Z_2)$ and $\mathrm{Ext}_{\mathfrak{a}}(H^*(bsp), Z_2)$ are easily computed as in [8; Section 1].

LEMMA 3. *The map* $bo \xrightarrow{\theta} \Sigma^4 bsp$ *satisfies* $\theta^*(\iota_4) = Sq^4(\iota_0)$, *where* ι_4 *and* ι_0 *generate* $H^4(\Sigma^4 bsp)$ *and* $H^0(bo)$, *respectively.*

Proof. This is proved as [8; Lemma 3.4]. We give a more elementary proof. If Lemma 3 were not true, then $\theta^*(\iota_4) = 0$, and so there would exist a short exact sequence of $\mathfrak{a}$-modules

$$0 \to \mathfrak{a}//\mathfrak{a}_1 \to H^*(bJ) \to s^3\mathfrak{a}/\mathfrak{a}(Sq^1, Sq^5) \to 0,$$

(where s^i denotes the increase of degrees by i), and hence a long exact sequence in $\mathrm{Ext}_{\mathfrak{a}}(\ \ , Z_2)$. This would imply $\mathrm{Ext}_{\mathfrak{a}}^{s,s+3}(H^*(bJ), Z_2) = Z_2$ for $s = 0, 1, 2, 3$, and the Adams spectral sequence converging to $\pi_*(bJ)$ would imply that 16 divides the order of $\pi_3(bJ)$, contradicting Proposition 2. ■

Let R_{Sq^4} denote right multiplication by Sq^4 and let $K = ker(s^4\mathfrak{a}/\mathfrak{a}(Sq^1, Sq^5) \cdot \xrightarrow{R_{Sq^4}} \mathfrak{a}//\mathfrak{a}_1)$. Since the cokernel of this homomorphism is $\mathfrak{a}//\mathfrak{a}_2$, we obtain a short exact sequence

$$0 \to \mathcal{A}//\mathcal{A}_2 \to H^*(bJ) \to s^{-1}K \to 0 \tag{1}$$

Since Sq^1Sq^4, Sq^7Sq^4, and $(Sq^4Sq^6 + Sq^7Sq^3)Sq^4$ lie in $\mathcal{A}(Sq^1, Sq^5)$, and $Sq^4Sq^4 \in \mathcal{A}(Sq^1, Sq^2)$, there is a homomorphism

$$R_{Sq^4}: s^8\mathcal{A}/\mathcal{A}(Sq^1, Sq^7, Sq^4Sq^6 + Sq^7Sq^3) \to K. \tag{2}$$

To show this is an isomorphism, let
$I = \text{image } (R_{Sq^4}: s^4\mathcal{A}/\mathcal{A}(Sq^1, Sq^5) \to \mathcal{A}//\mathcal{A}_1)$. There are short exact sequences of $\mathcal{A}$-modules

$$0 \to I \to \mathcal{A}//\mathcal{A}_1 \to \mathcal{A}//\mathcal{A}_2 \to 0$$

$$0 \to K \to s^4\mathcal{A}/\mathcal{A}(Sq^1, Sq^5) \to I \to 0$$

and applying $\mathrm{Ext}_{\mathcal{A}}(\ \ , Z_2)$ yields long exact sequences

$$\to \mathrm{Ext}_{\mathcal{A}_2}{}^{s,t}(Z_2, Z_2) \xrightarrow{\phi} \mathrm{Ext}_{\mathcal{A}_1}{}^{s,t}(Z_2, Z_2) \to \mathrm{Ext}_{\mathcal{A}}{}^{s,t}(I, Z_2) \to \mathrm{Ext}_{\mathcal{A}_2}{}^{s+1,t}(Z_2, Z_2) \to$$

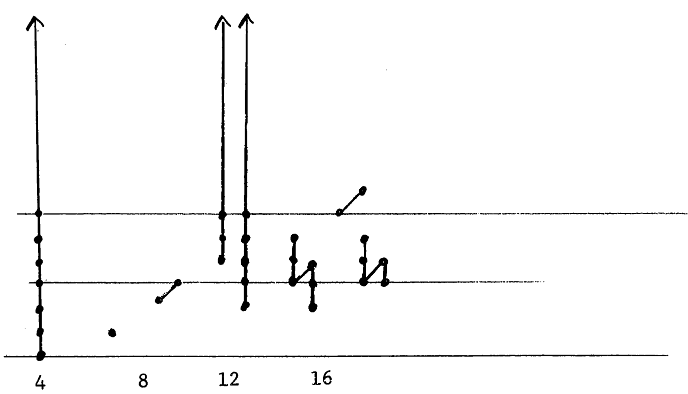
and

$$\to \mathrm{Ext}_{\mathcal{A}}{}^{s,t}(I, Z_2) \xrightarrow{\psi} \mathrm{Ext}_{\mathcal{A}}{}^{s,t}(s^4\mathcal{A}/\mathcal{A}(Sq^1, Sq^5), Z_2) \to \mathrm{Ext}_{\mathcal{A}}{}^{s,t}(K, Z_2) \to .$$

The image of ϕ consists of the elements of $\mathrm{Ext}_{\mathcal{A}_1}{}^{s,t}(Z_2, Z_2)$ for which $t - s \not\equiv 4(8)$. Thus $\mathrm{Ext}_{\mathcal{A}}(I, Z_2)$ is easily described in terms of $\mathrm{Ext}_{\mathcal{A}_2}(Z_2, Z_2)$; it begins as in Table 3. By low-level minimal resolution computations together with the compatibility of ψ with Yoneda multiplication by the periodicity element ω (see [2]), one shows that the image of ψ consists of the elements for which $t - s \not\equiv 0(8)$. Thus $\mathrm{Ext}_{\mathcal{A}}{}^{s,t}(K, Z_2)$ is $\mathrm{Ext}_{\mathcal{A}_2}{}^{s+2,t}(Z_2, Z_2)$ without $\omega^i x^{s,t}$ for all $x^{s,t}$ such that $t - s \leq 3$, without $\omega^i c_0$ and $\omega^i h_1 c_0$, where c_0 is the nonzero element with bi-

$t - s$

Table 3: $\mathrm{Ext}_{\mathcal{A}}{}^{s,t}\ (I, Z_2)$



degree (3, 11), and with infinite towers built upon $\omega^i h_2^2$. In particular

$$\mathrm{Ext}_{\mathfrak{a}}^{0,t}(K, Z_2) \approx \begin{cases} Z_2 & t = 8 \\ 0 & t \neq 8 \end{cases} \quad \text{and } \mathrm{Ext}_{\mathfrak{a}}^{1,t}(K, Z_2) \approx \begin{cases} Z_2 & t = 9, 15, 18 \\ 0 & \text{otherwise.} \end{cases}$$

Thus K is an $\mathfrak{A}$-module on one generator and three relations; it is easily verified that R_{Sq^4} in (2) sends generator to generator and relation to relation and hence is an isomorphism.

Thus (1) becomes

$$0 \to \mathfrak{A}//\mathfrak{A}_2 \to H^*(bJ) \to s^7\mathfrak{A}/\mathfrak{A}(Sq^1, Sq^7, Sq^4Sq^6 + Sq^7Sq^3) \to 0 \tag{3}$$

and its long exact $\mathrm{Ext}_{\mathfrak{a}}(\ , Z_2)$-sequence shows that

$$\mathrm{Ext}_{\mathfrak{a}}^{0,t}(H^*bJ, Z_2) = \begin{cases} Z_2 & t = 0, 7 \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$\mathrm{Ext}_{\mathfrak{a}}^{1,t}(H^*bJ, Z_2) = \begin{cases} Z_2 & t = 1, 2, 4, 8, 14, 17 \\ 0 & \text{otherwise.} \end{cases} \quad \text{Using this together}$$

with (3) shows that $H^*(bJ)$ has generators g_0 and g_7 with the only relations being $Sq^1 g_0$, $Sq^2 g_0$, $Sq^4 g_0$, $Sq^1 g_7 + \theta_8 g_0$, $Sq^7 g_7 + \theta_{14} g_0$, and $(Sq^4Sq^6 + Sq^7Sq^3)g_7 + \theta_{17} g_0$, where $\theta_8 \in (\mathfrak{A}//\mathfrak{A}_2)_8 = \{0, Sq^8\}$, $\theta_{14} \in (\mathfrak{A}//\mathfrak{A}_2)_{14} = \{0, Sq^{14}\}$, and $\theta_{17} \in (\mathfrak{A}//\mathfrak{A}_2)_{17} = \{0\}$. $\theta_{14} = 0$ because $Sq^1Sq^7 = 0$ but $Sq^1Sq^{14} \neq 0 \in \mathfrak{A}//\mathfrak{A}_2$. If $\theta_8 = 0$, then there would be an isomorphism $\mathrm{Ext}_{\mathfrak{a}}^{s,t}(H^*bJ, Z_2) \approx \mathrm{Ext}_{\mathfrak{a}}^{s,t}(Z_2, Z_2) \oplus \mathrm{Ext}_{\mathfrak{a}}^{s,t}(s^7\mathfrak{A}/\mathfrak{A}(Sq^1, Sq^7, Sq^4Sq^6 + Sq^7Sq^3))$ and then the Adams spectral sequence would imply that 32 divides the order of $\pi_7(bJ)$, contradicting Proposition 2; hence $\theta_8 = Sq^8$, proving part (i).

To prove part (ii) it remains to compute the boundary homomorphisms $\mathrm{Ext}_{\mathfrak{a}_2}^{s-1,t}(Z_2, Z_2) \xrightarrow{d} \mathrm{Ext}_{\mathfrak{a}}^{s,t}(s^7\mathfrak{A}/\mathfrak{A}(Sq^1, Sq^7, Sq^4Sq^6 + Sq^7Sq^3))$. By inspection the only possible elements not in the kernel of d are $h_0^k\omega^{i+1}$ $(i \geq 0)$. We shall show below that $d(h_0^k\omega^{i+1})$ is nonzero if and only if i is even, proving part *(ii)*.

Sq^1 acts as a differential on an $\mathfrak{A}$-module M, so that we can define $H_*(M; Sq^1)$.

LEMMA 4. *There is a* $1 - 1$ *correspondence between infinite towers in* $\mathrm{Ext}_{\mathfrak{a}}^{*,t}$. (M, Z_2) *and a basis for* $H_t(M; Sq^1)$.

Proof. We define an epimorphism of $\mathfrak{A}$-modules $N \xrightarrow{\phi} M$ inducing an isomorphism $H_*(N; Sq^1) \xrightarrow{\phi_*} H^*(M; Sq^1)$ by letting $N = \oplus\,\mathfrak{A} \oplus\ \oplus\,\mathfrak{A}//\mathfrak{A}_0$, where the first sum corresponds to (and the generators map to) a set of $\mathfrak{A}$-generators of M, and the second sum corresponds to (and the generators map to) a basis for $H_*(M; Sq^1)$. Let $L = ker(\phi)$; then $H_*(L, Sq^1) = 0$, so by [2; Theorem 2.1] $\mathrm{Ext}_{\mathfrak{a}}^{s,t}(L, Z_2) = 0$ if $3s \geq t + 6$. Thus $\mathrm{Ext}_{\mathfrak{a}}^{s,t}(M, Z_2) \to \mathrm{Ext}_{\mathfrak{a}}^{s,t}(N, Z_2)$ is an isomorphism for $3s \geq t + 6$.

But $\mathrm{Ext}_{\mathfrak{a}}^{s,t}(\mathfrak{1}, Z_2) = \begin{cases} Z_2 & s = t = 0 \\ 0 & \text{otherwise} \end{cases}$ and

$\mathrm{Ext}_{\mathfrak{a}}^{s,t}(\mathfrak{A}//\mathfrak{A}_0, Z_2) = \begin{cases} Z_2 & t = s \\ 0 & \text{otherwise,} \end{cases}$ so the Lemma follows. ∎

Let $Sq(i_1, \cdots)$ denote elements in the Milnor basis [9] and χ denote the canonical antiautomorphism [9]. By computing in $\chi((\mathfrak{A}//\mathfrak{A}_2)^*)$ as in [4; Section 6], we find that a basis for $H_*(\mathfrak{A}//\mathfrak{A}_2; Sq^1)$ consists of all $\chi(Sq(8i, 4j))$ and a basis for $H_*(\mathfrak{A}/\mathfrak{A}(Sq^1, Sq^7, Sq^4Sq^6 + Sq^7Sq^3); Sq^1)$ consists of $\chi(Sq(8i) + Sq(8i - 6, 2))$ and $\chi(Sq(8i + 6, 4j) + Sq(8i, 4j + 2))$. For example,

$$Sq^1(\chi(Sq(8i) + Sq(8i-6,2))) = \chi(Sq(8i-6))Sq^7 + (\chi(Sq(8i) + Sq(8i-6,2)))Sq^1$$

because $Sq(8i)Sq^1 + Sq(8i-6, 2)Sq^1 = \chi(Sq^7)Sq(8i-6) + Sq^1(Sq(8i) + Sq(8i-6,2))$.

Under the correspondence of Lemma 4, the tower $h_0^k\omega^{i+1}$ corresponds to $\chi(Sq(8i + 8))$. Hence $d(h_0^k\omega^{i+1})$ is nonzero if and only if the tower is not present in $\mathrm{Ext}_{\mathfrak{a}}(H^*bJ, Z_2)$ if and only if $\chi(Sq(8i + 8))g_0 \in im(Sq^1)$ if and only if $\chi(Sq(8i+8))g_0 = Sq^1(\chi(Sq(8i) + Sq(8i-6,2)))g_7$.

The above example shows that $Sq^1(\chi(Sq(8i) + Sq(8i - 6, 2)))g_7 = \chi(Sq(8i) + Sq(8i-6,2))Sq^1g_7 = \chi(Sq(8i) + Sq(8i-6,2))Sq^8g_0$. Thus to show d is as claimed it is equivalent to show $\chi(Sq(8i) + Sq(8i-6,2))Sq^8 = Sq(8i + 8) +$ other Milnor basis elements if and only if i is even. But this follows easily since

$$\begin{aligned} &\langle \xi_1^{8i+8}, \chi(Sq(8i) + Sq(8i-6,2))Sq^8\rangle \\ &= \tbinom{8i+8}{8} \langle \xi_1^{8i}, \chi(Sq(8i) + Sq(8i-6,2))\rangle \\ &= \tbinom{8i+8}{8} \langle \chi(\xi_1)^{8i}, Sq(8i) + Sq(8i-6,2)\rangle = \binom{8i+8}{8} \end{aligned}$$

which is a nonzero element of Z_2 if and only if i is even.

Let $\overline{bJ}$ denote the cofibre of the map $S^0 \to bJ$. $\pi_*(\overline{bJ})$ is the subgroup of the 2-primary stable homotopy of spheres complementary to the image of the J-homomorphism (plus the Adams elements μ_r [3; 1.3]). By techniques similar to those used in proving Theorem 1 we can prove.

THEOREM 5. *$H^*(bJ)$ has minimal generating set g_7 and g_{2^n} $(n \geq 4)$ and minimal set of relations $Sq^2Sq^1g_7$, Sq^7g_7, $Sq^8Sq^1g_7$, $(Sq^4Sq^6 + Sq^7Sq^3)g_7$ and $R(i, j)$* $(0 \leq i < j - 1$ or $i = j, j \geq 4)$, *where $R(i, j)$ corresponds to the Adem relation for $Sq^{2^i}Sq^{2^j}$,* with the final Sq^{2^k} in each term replaced by

$$\begin{cases} 0 & k = 0, 1, 2 \\ Sq^1g_7 & k = 3 \\ g_{2^k} & k \geq 4. \end{cases}$$

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References

[1] J. F. Adams, *Vector fields on spheres*, Ann. Math. **75** (1962), 603–32.

[2] J. F. Adams, *A periodicity theorem in homological algeqra*, Proc. Cambr. Phil. Soc. **62** (1966), 365–77.

[3] J. F. Adams, *On the qroups* J(X)-IV, Topology **5** (1966), 21–71.

[4] D. Anderson, E. Brown and F. P. Peterson, *The structure of the spin cobordism ring*, Ann. Math. **86** (1967), 271–98.

[5] D. M. Davis and M. Mahowald, *A strong non-immersion theorem for* RP^{8l+7}, Bull. Amer. Math. Soc. **81** (1975), 155–56.

[6] A. Iwai and N. Shimada, *On the cohomology of some Hopf algebras*, Nagoya Math. J. **30** (1967), 103–11.

[7] M. E. Mahowald, *The order of the image of the J-homomorphism*. Bull. Amer. Math. Soc. **76** (1970), 1310–13.

[8] R. J. Milgram, *The Steenrod algebra and its dual for connective K-theory*. Proc Homotopy Conf., Evanston, Ill. 1974, Monograph of the Mexican Mathematical Society.

[9] J. Milnor, *The Steenrod algebra and its dual*, Ann. of Math **67** (1958), 150–71.

[10] R Stong, *Determination of* $H^*(BO(k,\ \infty);\ Z_2)$, Trans. Amer Math. Soc. **107** (1963), 526–44.