AN ANALYTIC FOLIATION OF THE PLANE WITHOUT WEAK FIRST INTEGRALS OF CLASS C^1

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1. Introduction

T. Wazewski gave in [1] an example of a smooth foliation of the plane without non trivial weak first integrals of class C^1. However, his method did not allow the solution of the problem in the analytic case. Here we answer this question with an explicit construction proving:

Theorem. *There exists a real analytic structure of the plane and an analytic foliation of this plane such that:*
1) *the branch leaves form an everywhere dense set*
2) *every function of class C^1 which is constant on every leaf is globally constant.*

The example given proves in particular the existence of an analytic structure on the "compound feather" ("plume composée," in [2]) and the existence of a Hausdorff analytic line bundle over this space where the non separate points are everywhere dense.

First we shall construct a simply connected open set U in the plane such that the foliation defined by $dy = 0$ on U has the property 1). By the conformal representation theorem we obtain a foliation of the plane which is analytic for the usual structure and verifies 1). Then we shall define a new real analytic structure on U such that the foliation is also analytic for this structure and verifies moreover 2).

2. Construction of the open set U

Let $(y_n)_{n\in N}$ be a numeration of the rational numbers. We define a sequence $(F_n)_{n\in N}$ of closed sets of $\boldsymbol{R}^2$ by induction:
$F_0 = [-\frac{1}{2}, \frac{1}{2}] \times [y_0, +\infty[$

By induction, F_n is a union of vertical closed half-bands of width $1/3^n$ and F_{n+1} is obtained from F_n by the following: let

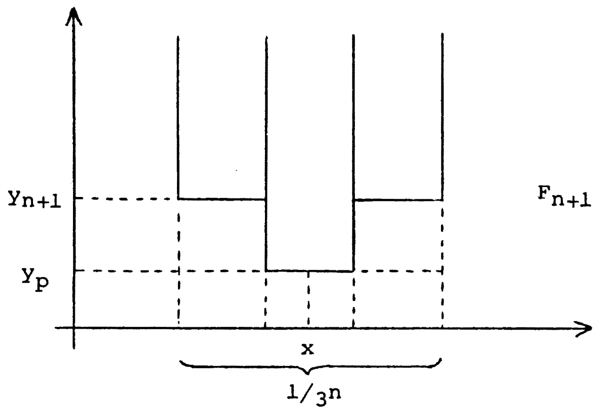
$$B = \left[x - \frac{1}{3^n\cdot 2}, x + \frac{1}{3^n\cdot 2}\right] \times [y_p, +\infty[\qquad (p \leq n)$$

be one of the half-bands composing F_n.

1) If $y_{n+1} > y_p$, B contains three half-bands of F_{n+1} of equal width $1/3^{n+1}$: the lateral ones are constructed over the ordinate y_{n+1} and the middle one over y_p.

2) If $y_{n+1} < y_p$, B contains one half-band of F_{n+1}, the half-band

$$\left[x - \frac{1}{3^{n+1}\cdot 2}, x + \frac{1}{3^{n+1}\cdot 2}\right] \times [y_p, +\infty[$$



3) If there exists $q \in \{0, 1, \cdots, n\}$ such that $y_{n+1} > y_q$, then $F_{n+1} \subset F_n$.
4) If $y_{n+1} < y_q$ for every $q \in \{0, 1, \cdots, n\}$, then F_{n+1} contains two half-bands in the complement of F_n:

$$F_{n+1} - F_n = \left(\left[n + 1 - \frac{1}{3^{n+1}\cdot 2}, n + 1 + \frac{1}{3^{n+1}\cdot 2}\right] \cup \left[-n - 1 - \frac{1}{3^{n+1}\cdot 2}, -n - 1 + \frac{1}{3^{n+1}\cdot 2}\right]\right) \times [y_{n+1}, \infty[.$$

Properties of the sequence $(F_n)_{n \in N}$

1) For every bounded set $K \subset \boldsymbol{R}^2$, there exists $n_0 \in N$ such that the sequence $(F_n \cap K)_{n \geq n_0}$ is decreasing. Hence $(F_n)_{n \in N}$ converges to a closed set F.

2) Let B be one of the half-bands composing F_n. The only point of the boundary of B which is in F is the point situated in the middle of the horizontal part of this boundary.

3) By 2) the interior of the projection of F on the horizontal axis is empty, hence the interior of F is empty.

4) The complement U of F in the plane is simply connected because F is a union of half-lines.

We consider now the foliation induced on U by the horizontal lines $y =$ constant. We shall say that $A \subset U$ is saturated if it is a union of leaves.

5) Every leaf is bounded because there exists a subsequence $(y_{n_p})_{p \in N}$ such that for every p, $y_{n_p} < y_q$ for every $q < n_p$, it converges to $-\infty$ and F contains the half-lines $\{-n_p\} \times [y_{n_p}, +\infty[$ and $\{n_p\} \times [y_{n_p}, +\infty[$.

6) Let $L =]a, b[\times \{y\}$ and $L' =]a', b'[\times \{y'\}$ be two leaves with $]a, b[\cap]a', b'[\neq \varnothing$. Let us suppose that $y' > y$. Hence L' intersects $]a, b[\times [y, +\infty[$. But (a, y) and (b, y) are in F, hence the half-lines $\{a\} \times$

$[y, +\infty[$ and $\{b\} \times [y, +\infty[$ are contained in F. This shows that $L' \subset]a, b[\times [y, +\infty[$, hence the projection on the horizontal axis of two distinct leaves are either disjoint or contained one in other.

We shall say that a leaf L is *under* a leaf L' (or L' is *over* L) if the projection of L contains the projection of L'.

7) *Definition.* Two leaves are *non separated* if the corresponding points in the space of the leaves are non separated. A *branch leaf* is a leaf which is non separated from another leaf.

Let L_1 and L_2 be two non separated leaves. Then L_1 and L_2 have the same ordinate. If $L =]a, b[\times \{y\}$ is a leaf distinct from L_1 and under L_1, then L is also under L_2 ($\{a, b\} \times [y, \infty[\subset F$, hence $]a, b[\times [y, \infty[\cap U$ is a closed saturated neighborhood of L_1, which implies that it contains also L_2).

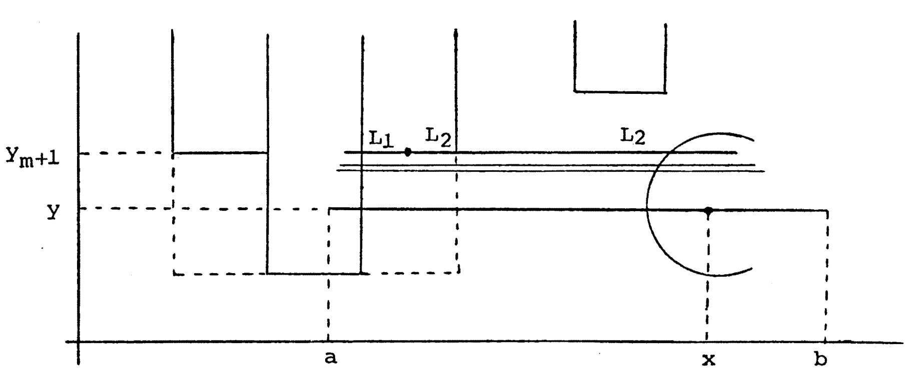
3. The union of the branch leaves is everywhere dense in U

Let $(x, y) \in U$ and $]a, b[\times \{y\}$ the leaf containing (x, y). There exists $n_0 \in N$ such that (a, y) and (b, y) are in F_n for every $n \geq n_0$. For every $\epsilon > 0$ let $m \in N$ verifying the following conditions:

1. $m \geq n_0$
2. $2/3^m < b - x$ and $2/3^m < x - a$
3. $y < y_{m+1} < y + \epsilon$
4. $y_{m+1} < y_p$ for every $p \leq m$ such that $y_p > y$

Such a number m does exist: let m_0 be the smallest integer greater than n_0 and verifying 2. It is sufficient to take the smallest integer greater or equal to m_0 such that $y_{m+1} \in]y, y + \eta[$, where $\eta = inf\{\epsilon, y_i - y; i \leq m_0 \text{ with } y_i > y\}$

Let B be the half-band of F_m containing (a, y). At the order $m + 1$, by the above conditions, B gives three half-bands for F_{m+1}, the third one gives two non separated leaves (L_1 and L_2) by property 2), and one of these leaves intersects the disk centered in (x, y) of radius ϵ (by condition 4. the other half-bands of F_m between those which contain (a, y) and (b, y) are constructed over rational numbers greater than y_{m+1}). Hence every neighborhood of (x, y) intersects a branch leaf.



4. Construction of a convenient analytic structure on U

We shall construct by induction an increasing sequence $(V_n)_{n\in N}$ of open sets converging to U, with a convenient analytic atlas on every V_n.

Let D_0 be a vertical line contained in U, V_0 the saturation of D_0 and $\varphi_0 : V_0 \to \boldsymbol{R}^2$ the inclusion. By property 6), if $(x, y) \in V_0$ then $\{x\} \times]-\infty, y] \subset V_0$. Hence the boundary G_0 of V_0 in U is a "stair" in the following sense: if L and L' are two leaves of G_0, then their projections on the horizontal axis are disjoint. We can note that G_0 is the union of all the leaves which are non separated from some leaf of V_0.

Let us suppose that V_n is constructed. Let G_n be the boundary of V_n in U. For every leaf L contained in G_n, let D be a vertical line contained in U which intersects L in its middle third (it exists by property 3)). Let $D_{n,L}$ be the connected component of $D - (G_0 \cup \cdots \cup G_{n-1})$ which intersects L, $V_{n,L}$ the saturation of $D_{n,L}$ and $\varphi_{n,L} : V_{n,L} \to \boldsymbol{R}^2$ defined by $\varphi_{n,L}(x, z) = (x, (z - y)^3)$ where y is the ordinate of L.

Let $V_{n+1} = V_n \bigcup_{L\subset G_n} V_{n,L}$. By induction, for every $n \in N$, V_n is open and it is the saturation of a union of vertical lines (hence so is $V = \bigcup_n V_n$) and G_n is a "stair," union of the leaves which are not in V_n but are non separated from some leaf of V_n.

If $(x, y) \in U - V$, $\Delta = \{x\} \times]-\infty, y[$ intersects every G_n and $\Delta \cap (\bigcup_n G_n)$ is infinite (the G_n are disjoint). By property 6), all the leaves of $G = \bigcup_n G_n$ which intersect Δ are larger than the leaf containing (x, y). But by the construction ($D_{n,L}$ intersects L in its middle third) and by property 7), if a leaf L of G_n is over a leaf L' of G_m (hence $n > m$) then the length of L is inferior to two thirds of the length of L', hence the limit inferior of the lengths of the leaves of G intersecting Δ must be zero, which is impossible.

Clearly $\{(V_0, \varphi_0), (V_{n,L}, \varphi_{n,L}); n \in N, L \subset G_n\}$ is a real analytic atlas on U: the possible singularities must occur on G, but if $(n, L) \neq (n', L')$ then $V_{n,L} \cap V_{n',L'} \cap G = \varnothing$ (because $V_{n,L} \cap G = L$).

5. The first integrals of class C[1]

In the following, U has the analytic structure just constructed and U^* is the same open set with the usual one (induced by that of $\boldsymbol{R}^2$).

Let $f : U \to \boldsymbol{R}$ be a C^1 function which is constant on every leaf. The identity $U^* \to U$ is analytic, hence $f : U^* \to \boldsymbol{R}$ is also C^1.

Let L and L' be two non separated leaves of ordinate y with $L \subset G$ and $L' \not\subset G$. Let $(V_{n,L_1}, \varphi_{n,L_1})$ be a chart with $L' \subset V_{n,L_1}$ (hence $L_1 \neq L'$) and $(V_{m,L}, \varphi_{m,L})$ the (unique) chart containing L.

f is C^1 on U, hence $f \circ \varphi_{m,L}^{-1} : (u, v) \to f(u, \sqrt[3]{v} + y)$ is C^1 near $\varphi_{m,L}(L)$, hence $f : U^* \to \boldsymbol{R}$ is of rank zero on L. But f is constant on the leaves and C^1 on U^*, hence the rank of f in U^* is also zero on L' (by property 7)). The identity $U \to U^*$ is analytic on a neighborhood of L' (because $L' \not\subset G$). Hence $f : U \to \boldsymbol{R}$ is of rank zero on L'.

This shows that f is of rank zero on every branch leaf which is not in G. But the union of these leaves is dense in U because G is closed with $\mathring{G} = \emptyset$ and by section 3. Hence f is constant.

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