A MODEL OF BRANCHING PROCESSES WITH RANDOM ENVIRONMENTS

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0. Introduction

Smith and Wilkinson (1969) have formulated a model for a branching process with random environments. Their model may be described as follows. Let $\{\eta_i, i = 0, 1, 2, \cdots\}$ be a sequence of independent identically distributed (i.i.d.) random variables taking values in some space Θ. Associated with each $\theta \in \Theta$ there is a probability generating function (p.g.f.)

$$f_\theta(s) = \sum_{j=0}^{\infty} p_j(\theta)s^j.$$

For each realization of the process $\{\eta_i, i = 0, 1, 2, \cdots\}$ there evolves a population Z_n, $n = 0, 1, 2, \cdots$, in the following way. Suppose the zero-th generation consists of Z_0 objects. Each one of these objects, independently of the others, creates offspring according to the p.g.f. $f_{\eta_0}(s)$, forming in this way the Z_1 objects of the first generation, i.e.,

$$Z_1 = \sum_{i=1}^{Z_0} X_{1,i}$$

where $X_{1,1}, X_{1,2}, \cdots, X_{1,Z_0}$ are i.i.d. random variables with p.g.f. $f_{\eta_0}(s)$. The second generation consists of the progeny of the Z_1 objects in the first generation, and its number Z_2 is given by

$$Z_2 = \sum_{i=1}^{Z_1} X_{2,i}$$

where $X_{2,1}, X_{2,2}, \cdots, X_{2,Z_1}$ are i.i.d. random variables with p.g.f. $f_{\eta_1}(s)$. The process continues in this way. In this model, thanks to the fact that the "environmental process" $\{\eta_i, i = 0, 1, 2, \cdots\}$ consists of i.i.d. random variables, the process $\{Z_n, n = 0, 1, 2, \cdots\}$ just described is a Markov process.

In order to generalize Smith and Wilkinson's model, Athreya and Karlin (1971) considered a model described similarly to that of Smith and Wilkinson, except that the environmental process $\{\eta_i, i = 0, 1, 2, \cdots\}$ is allowed to be more general, e.g. a stationary ergodic process or a Markov chain.

Both of these models are for a discrete time environmental process. Kaplan (1973) considered a branching process with random environments, in which the environment changes continuously in time according to a continuous time stationary ergodic process.

In this paper we consider a continuous time branching process with a random environment, in which the environment changes according to a continuous time Markov chain $v(t)$. More specifically, suppose there are Z_0 particles at time 0, and that these particles start branching following a law of reproduction associated with the initial state $v(0)$ of the chain. When the chain jumps to some other state, then the existing particles at the time of the jump change their law of reproduction to another one associated with the new state of the chain. Proceeding in this manner, the particles change their law of reproduction when the chain $v(t)$ jumps, and this law is associated with the state of the chain at that time. The idea of considering this model is based on the concept of a random evolution.

Random evolutions were defined by Griego and Hersh (1969), roughly as follows. Assume we are given an n-state continuous time Markov chain $v(t)$, and a system which evolves in time with n different possible laws of evolution. When $v(t)$ is in state i, the system evolves according to its i-th law of evolution, and changes its law of evolution to the j-th one when $v(t)$ jumps to state j. For a comprehensive study of the theory of random evolutions and its development, the reader may refer to Griego and Hersh (1971), Hersh (1974) and Pinsky (1974).

As we can see, the concept of a random evolution and that of a branching process with a random environment considered in this paper are very similar. So, it can be expected that one can use results and techniques of random evolutions to answer some questions concerning branching processes with random environments.

In this paper we study:

i) Extinction probabilities,

ii) Expected size of the population at time t, and

iii) Limit theorems.

1. Notation and preliminaries

Branching Processes.

We consider only branching processes with stationary transition probabilities. Following Harris (1963) we have the following definition.

Definition 1.1. A continuous time Markov branching process is a Markov chain $\{X(t),\ t \geq 0\}$ whose transition probabilities $\mathcal{P}_{ij}(t)$ are a solution of the forward equations

$$\frac{d\mathcal{P}_{i,k}(t)}{dt} = -kb\mathcal{P}_{i,k}(t) + b\sum_{j=1}^{k+1} j\mathcal{P}_{i,j}(t)p_{k-j+1}\,, \qquad \mathcal{P}_{i,k}(0+) = \delta_{i,k}\,,$$

where b is a positive real number, and the p_i are non-negative real numbers satisfying $p_1 = 0$ and $\sum_{i=0}^{\infty} p_i = 1$. $\delta_{i,k}$ is the Kronecker delta function defined by

$$\delta_{i,k} = \begin{cases} 0 & \text{if} \quad i \neq k \\ 1 & \text{if} \quad i = k. \end{cases}$$

Now, let us define $f(s) = \sum_{j=0}^{\infty} p_j s^j$, $|s| \leq 1$ and $h(s) = b(f(s) - s)$. Also let $F_i(s,\ t) = \sum_{j=0}^{\infty} \mathcal{P}_{i,j}(t)s^j$ be the probability generating function of the process $\{X(t),\ t \geq 0\}$, satisfying $X(0) = i$. $F_1(s,\ t)$ will be denoted just by $F(s,\ t)$. Let

$$\lambda = \frac{dh(s)}{ds}\Big|_{s=1}\,,$$

then

$$E[X(t) \mid X(0) = 1] = \frac{\partial}{\partial s} F(s, t) \mid_{s=1} = e^{\lambda t}.$$

(See [7], chap. VII, sect. 6).

Random Evolutions. Griego and Hersh (1969) introduced the concept of random evolution in the following way. Let $\{v(t), t \geq 0\}$ be a continuous time stationary Markov chain with state space $\{1, \cdots, N\}$, and infinitesimal transition probability matrix $Q = (q_{ij})$. Let $\{T_i(t), t \geq 0\}$, $i = 1, \cdots, N$, be a family of strongly continuous semigroups of bounded linear operators defined on a Banach space $\mathfrak{B}$. Let τ_j be the time of the j-th jump of $v(t)$, and $N(t)$ be the number of jumps of the chain before time t.

Definition 1.2. The random evolution $M = \{M(t), t \geq 0\}$ associated with the semigroups $\{T^{(j)}(t), t \geq 0\}$, $j = 1, \cdots, N$, and with the Markov chain $v(t)$ s defined by

$$M(t) = T^{(v(0))}(\tau_1)T^{(v(\tau_1))}(\tau_2 - \tau_1) \cdots T^{(v(\tau_{N(t)}))}(t - \tau_{N(t)}).$$

Let $\tilde{\mathfrak{B}}$ be the N-fold cartesian product of $\mathfrak{B}$ with itself. An element in $\tilde{\mathfrak{B}}$ will be denoted by $\tilde{f} = (f_1, \cdots, f_N)$. We equip $\tilde{\mathfrak{B}}$ with any appropriate norm so that $\| \tilde{f} \| \rightarrow 0$ as $\| f_j \| \rightarrow 0$, $j = 1, 2, \cdots, N$. A semigroup $\tilde{T}(t)$ on $\tilde{\mathfrak{B}}$ is defined componentwise by $(\tilde{T}(t)\tilde{f})_j = E_j[M(t)f_{v(t)}]$, where E_j denotes expectation with respect to the probability P_j which assigns probability 1 to the set of sample paths of $\{v(t), t \geq 0\}$ which satisfy $v(0) = j$. Griego and Hersh called the semigroup $\{\tilde{T}(t), t \geq 0\}$, the "expectation semigroup" of M, and they proved that $\tilde{U}(t) = \tilde{T}(t)\tilde{f}$ solves

$$\frac{\partial U_j}{\partial t} = A^{(j)} U_j + \textstyle\sum_{k=1}^{N} q_{jk} U_k, \qquad U_j(0+) = f_j, \tag{1.1}$$

for each $j = 1, \cdots, N$, where $A^{(j)}$ is the infinitesimal generator of $T^{(j)}$. This result is an operator-theoretical version of the classical Feynman-Kac formula, and we will refer to it as the Feynman-Kac formula.

Branching processes with global environmental changes (BPGEC).

Let $\{X^{(k)}(t), t \geq 0\}$, $k = 1, 2, \cdots, N$, be N continuous time Markov branching processes. We assume that for each one of these processes we have defined the different parameters associated with them, that are described before. We will use superscripts to distinguish the parameters belonging to one process from those of another, e.g., $F_i^{(k)}(s, t)$, $i = 0, 1, 2, \cdots$, denote the p.g.f. of $X^{(k)}(t)$, $P_{ij}^{(k)}(t)$ denote its transition probabilities, etc. Throughout this paper, we will assume $P_{1,0}^{(k)}(t) + P_{1,1}^{(k)}(t) < 1$ for $t \geq 0$ and all $k = 1, 2, \cdots, N$.

Let $v(t)$, $t \geq 0$, be a right-continuous Markov chain with stationary transition probabilities, and state space $\{1, 2, \cdots, N\}$. Let $Q = (q_{ij})$ be its matrix of infinitesimal transition probabilities. Define τ_i as the time of the i-th jump of $v(t)$, and $N(t)$ as the number of jumps of $v(t)$ before time t.

The model considered in this paper is described as follows. Let t and $Z(t)$

denote the total elapsed time and the number of particles in the population at time t, respectively. Assume there are Z_0 particles at time $t = 0$. Each one of these particles reproduces, independently of the others, according to the p.g.f. $F^{(v(0))}(s, t)$. They keep this law of reproduction until the chain $v(t)$ jumps to another state $v(\tau_1)$. Then *each one* of the particles existing at time τ_1 changes its law of reproduction to that given by the p.g.f. $F^{(v(\tau_1))}(s, t - \tau_1)$. Hence the change in the environment affects all the particles equally, i.e., the environmental change is *global*. Inductively, if there are $Z(\tau_n)$ particles at the time of the n-th jump of the chain, then each one of these particles starts branching following the law of reproduction given by the p.g.f. $F^{(v(\tau_n))}(s, t - \tau_n)$, and continues reproducing in this manner until the chain jumps to the state $v(\tau_{n+1})$. At that time, τ_{n+1}, all the existing particles, $Z(\tau_{n+1})$, change their law of reproduction to that given by the p.g.f. $F^{(v(\tau_{n+1}))}(s, t - \tau_{n+1})$.

Notice that the process $\{Z(t), t \geq 0\}$ is not, in general, a Markov process. However, the two-component process $\{(v(t), Z(t)), t \geq 0\}$ does constitute a Markov process.

This model just described in words can be defined rigorously by "piecing out" the different branching processes at the jump times of $v(t)$. This was done for random evolutions by Griego and Moncayo (1970).

Each one of the branching processes $\{X^{(j)}(t), t \geq 0\}$ is a Markov process, hence there is a semigroup $\{S^{(j)}(t), t \geq 0\}$ associated with it. This semigroup is defined on the space of bounded sequences $l_\infty = \{f:I \rightarrow \boldsymbol{R}, f \text{ bounded}\}$ by

$$(S^{(j)}(t)f)(i) = \sum_{k=0}^{\infty} P_{i,k}^{(j)}(t)f(k),$$

for $i \in I$. Here I denotes the set $\{0, 1, 2, 3, \cdots\}$, and $\boldsymbol{R}$ denotes the set of real numbers.

Let $\mathfrak{M}(t)$ be the random evolution associated with the semigroups $\{S^{(j)}(t), t \geq 0\}, j = 1, \cdots, N$, and with the Markov chain $v(t)$, i.e.,

$$\mathfrak{M}(t) = S^{(v(0))}(\tau_1)S^{(v(\tau_1))}(\tau_2 - \tau_1) \cdots S^{(v(\tau_{N(t)}))}(t - \tau_{N(t)}).$$

Griego and Moncayo (1970) showed that the Markov semigroup associated with the pieced process $\{(v(t), Z(t)), t \geq 0\}$ is equal to the expectation semigroup of the random evolution $\mathfrak{M}(t)$. In fact they proved that for a bounded real-valued function φ defined on $\{1, 2, \cdots, N\} \times I$, the equality

$$E_{(j,i)}[\varphi(v(t), Z(t))] = E_j[\mathfrak{M}(t)\varphi(v(t), i)] \tag{1.2}$$

holds. The expectation in the left hand side refers to the pieced process starting in state (j, i), and the expectation in the right hand side refers to the chain starting in state j.

Let $C^{(j)}$ denote the infinitesimal generator of $S^{(j)}(t), j = 1, \cdots, N$; by the Feynman-Kac formula (1.1), we know that

$$U_j(i, t) = E_j[\mathfrak{M}(t)\varphi(v(t), i)]$$

solves

$$\frac{\partial U_j}{\partial t} = C^{(j)}U_j(i, t) + \sum_{k=1}^N q_{jk}U_k(i, t), \tag{1.3}$$

$$U_j(i, 0+) = \varphi(j, i).$$

Let $G_i^{(j)}(s, t)$ denote the p.g.f. of $Z(t)$ under the conditions $v(0) = j$ and $Z(0) = i$. As usual, $G^{(j)}(s, t)$ will denote $G_1^{(j)}(s, t)$. Now, for fixed $s \in (0, 1]$, define $\varphi_s:\{1, 2, \cdots, N\} \times I \to \boldsymbol{R}$ by $\varphi_s(j, i) = s^i$. By (1, 2) we obtain

$$E_{(j,i)}[s^{Z(t)}] = E_j[\mathfrak{M}(t)s^i].$$

Hence $G_i^{(j)}(s, t) = E_j[\mathfrak{M}(t)s^i]$, and by (1.3), $G_i^{(j)}(s, t)$ satisfies

$$\frac{\partial G_i^{(j)}(s, t)}{\partial t} = C^{(j)}G_i^{(j)}(s, t) + \sum_{k=1}^N q_{jk} G_i^{(k)}(s, t), \tag{1.4}$$

$$G_i^{(j)}(s, 0+) = s^i.$$

It should be remarked that $C^{(j)}$ acts on $G_i^{(j)}(s, t)$ considering this as a function of i. More specifically, if $C^{(j)} = (a_{mn}^{(j)})_{m,n=0,1,2,\ldots}$, then

$$C^{(j)}G_i^{(j)}(s, t) = \sum_{n=0}^\infty a_{in}^{(j)}G_n^{(j)}(s, t). \tag{1.5}$$

The property that characterizes branching processes is that the particles reproduce independently of each other. Thus, if there are n particles at time $t = 0$, then the population evolves probabilistically as the combined sum of n populations, each with one initial parent. From this branching property we obtain $G_n^{(j)}(s, t) = (G^{(j)}(s, t))^n$.

Therefore, we can write (1.5) as

$$C^{(j)}G_i^{(j)}(s, t) = \sum_{n=0}^\infty a_{in}^{(j)}(G^{(j)}(s, t))^n. \tag{1.5$'$}$$

By substituting this in (1.4) and then letting $i = 1$, we obtain that $G^{(j)}(s, t)$ satisfies

$$\frac{\partial G^{(j)}(s, t)}{\partial t} = \sum_{n=0}^\infty a_{1n}^{(j)}(G^{(j)}(s, t))^n + \sum_{k=1}^N q_{jk}G^{(k)}(s, t) \tag{1.4$'$}$$

$$G^{(j)}(s, 0+) = s \qquad (j = 1, 2, \cdots, N).$$

In the proof of the Feynman-Kac formula, Griego and Hersh (1971) showed that the expectation semigroup satisfies a renewal equation. Namely, $U_j(i, t) = E_j[\mathfrak{M}(t)\varphi(v(t), i)]$ satisfies

$$U_j(i, t) = S^{(j)}(t)\varphi(j, i)e^{q_{jj}t} + \int_0^t S^{(j)}(r)\sum_{k\neq j} U_k(i, t - r)q_{jk}e^{q_{jj}r}\, dr. \tag{1.6}$$

By again choosing $\varphi_s(j, i) = s^i$, we obtain $U_j(i, t) = G_i^{(j)}(s, t)$,

$$S^{(j)}(t)\varphi_s(j, i) = \sum_{n=0}^\infty P_{i,n}^{(j)}(t)s^n,$$

and

$$\begin{aligned} S^{(j)}(r)G_i^{(j)}(s - r) &= \sum_{n=0}^\infty P_{i,n}^{(j)}(r)G_n^{(j)}(s, t - r) \\ &= \sum_{n=0}^\infty P_{i,n}^{(j)}(r)(G^{(j)}(s, t - r))^n. \end{aligned}$$

By substituting these in (1.6) and then letting $i = 1$, we obtain

$$G^{(j)}(s, t) = \sum_{n=0}^{\infty} P_{1,n}{}^{(j)}(t)s^n e^{q_{jj}t} + \sum_{k\neq j} \int_0^t \sum_{n=0}^{\infty} P_{1,n}{}^{(j)}(r)(G^{(k)}(s, t - r))^n q_{jk} e^{q_{jj}r}\, dr. \tag{1.7}$$

We should note that equations (1.4′) and (1.7) are two equivalent equations for $G^{(j)}(s, t)$. Both involve the infinitesimal transition probability matrix Q of the Markov chain $v(t)$, but the latter involves the semigroup $\{S^{(j)}(t), t \geq 0\}$ and the former involves the infinitesimal generator $C^{(j)}$ of $\{S^{(j)}(t), t \geq 0\}$, i.e., (1.7) is the integral equation corresponding to (1.4′). To conclude this section we note that equation (1.7) can be written as

$$G^{(j)}(s, t) = F^{(j)}(s, t)e^{q_{jj}t} + \sum_{k\neq j} \int_0^t F^{(j)}(G^{(k)}(s, t - r), r) q_{jk} e^{q_{jj}r}\, dr. \tag{1.7′}$$

2. Extinction probabilities

Let's consider the BPGEC $\{Z(t), t \geq 0\}$ described in the preceding section, and define

$$B = \{Z(t) = 0 \quad \text{for some} \quad t \geq 0\}, \quad \text{and}$$

$$\zeta_j = P[B \mid v(0) = j, Z(0) = 1], \qquad j = 1, 2, \cdots, N.$$

We refer to B as the set of eventual extinction, and to $\zeta_1, \zeta_2, \cdots, \zeta_N$ as the probabilities of eventual extinction.

If we define $B_t = \{Z(t) = 0\}$, and $\zeta_j(t) = P[B_t \mid v(0) = j, Z(0) = 1]$, then the sets B_t increase to B, and $\lim_{t\to\infty} \zeta_j(t) = \zeta_j$, $j = 1, 2, \cdots, N$.

On the other hand, the probability $\zeta_j(t)$ is also given by $\zeta_j(t) = G^{(j)}(0, t)$, hence $\zeta_j = \lim_{t\to\infty} G^{(j)}(0, t)$.

Now, letting $s \to 0$ in the renewal equation (1.7′), and using the dominated convergence theorem we obtain that $G^{(j)}(0, t)$, or, equivalently, $\zeta_j(t)$ satisfies

$$\zeta_j(t) = P_{1,0}{}^{(j)}(t)e^{q_{jj}t} + \sum_{k\neq j} \int_0^t F^{(j)}(\zeta_k(t - r), r) q_{jk} e^{q_{jj}r}\, dr. \tag{2.1}$$

From this, to obtain an equation for ζ_j all we have to do is let $t \to \infty$ in (2.1). To avoid degenerate cases we will assume $q_{ii} \neq 0$ for $i = 1, 2, \cdots, N$. Thus, by the dominated convergence theorem, we obtain from (2.1)

$$\zeta_j = \sum_{k\neq j} \int_0^\infty F^{(j)}(\zeta_k, r) q_{jk} e^{q_{jj}r}\, dr.$$

Summarizing, we have proved the following

THEOREM 2.1. *If the chain $v(t)$ does not have absorbing states, i.e., $q_{jj} < 0$ for $j = 1, 2, \cdots, N$, then the extinction probabilities $\zeta_j = P[Z(t) = 0$ for some $t \mid v(0) = j, Z(0) = 1]$ satisfy*

$$\zeta_j = \sum_{k\neq j} \int_0^\infty F^{(j)}(\zeta_k, r) q_{jk} e^{q_{jj}r}\, dr, \qquad j = 1, 2, \cdots, N. \tag{2.2}$$

This theorem has the following interesting

COROLLARY. *If $v(t)$ is irreducible, then, either $\zeta_j = 1$ for $j = 1, 2, \cdots, N$, or $\zeta_j < 1$ for $j = 1, 2, \cdots, N$.*

Proof. Suppose $\zeta_i < 1$ for some i. Then $F^{(j)}(\zeta_i, r) < 1$ for all $r \geq 0$ and all $j = 1, 2, \cdots, N$, and by (2.2),

$$\zeta_j = \sum_{k \neq j} \int_0^\infty F^{(j)}(\zeta_k, r) q_{jk} e^{q_{jj} r} \, dr$$
$$< \sum_{k \neq j} q_{jk} \int_0^\infty e^{q_{jj} r} \, dr = -\Big(\sum_{k \neq j} q_{jk}\Big) q_{jj}^{-1} = 1$$

Thus, if one of the ζ_j is less than 1, then all of them are less than 1. The hypothesis of irreducibility is needed to obtain the inequality. Q.E.D.

From this corollary, we can say that there are two possible cases concerning extinction. The process, independently of the initial environment, either dies with probability one or lives forever with positive probability. Theorem 2.2 gives a criterion which discriminates between the two possibilities. The method was suggested to us by Norman Kaplan and the proof is based on the following theorem, which we state without proof.

THEOREM A (Athreya and Karlin (1971), Theorem 4). *Concerning the model of Athreya and Karlin described in the Introduction, the following is true: suppose the environmental process $\{\eta_i, i = 0, 1, 2, \cdots\}$ is an irreducible, positive recurrent stationary Markov chain with state space $(\Theta, \mathcal{B})$. For each $\theta \in \Theta$, let*

$$f_\theta(s) = \sum_{k=0}^\infty p_k(\theta) s^k$$

be the p.g.f. associated with θ, and $\{P(B \mid \theta), \theta \in \Theta, B \in \mathcal{B}\}$, be the family of transition probabilities of the chain. Assume

$$P\Big[\sum_{j=0}^\infty j p_j(\eta_i) < \infty ; 0 \leq p_0(\eta_i) + p_1(\eta_i) < 1 \quad \text{for all} \quad i\Big] = 1.$$

Let $\tilde{P}$ be the unique stationary measure of $\{\eta_i, i = 0, 1, 2, \cdots\}$ i.e.,

$$\tilde{P}(B) = \int_\Theta P(B \mid \theta) \, d\tilde{P}(\theta) \quad \textit{for} \quad B \in \mathcal{B}.$$

Furthermore, assume

$$\int_\Theta \lvert \operatorname{Log}(1 - f_\theta(0)) \rvert \, d\tilde{P}(\theta) = \int_\Theta \lvert \operatorname{Log}(1 - p_0(\theta)) \rvert \, d\tilde{P}(\theta) < \infty,$$

and

$$\int_\Theta \operatorname{Log}(f_\theta'(1)) \, d\tilde{P}(\theta) < \infty.$$

Then the population becomes extinct with probability 1 if and only if

$$\int_\Theta \operatorname{Log}(f_\theta'(1)) \, d\tilde{P}(\theta) \leq 0.$$

Now, let's return to our model.

THEOREM 2.2. *Let $\lambda^{(j)}$ be the expectation parameter of the j-th branching process*

$\{X^{(j)}(t), t \geq 0\}$, *i.e.*,

$$E[X^{(j)}(t) \mid X^{(j)}(0) = 1] = \frac{\partial F^{(j)}(s, t)}{\partial s}\bigg|_{s=1} = e^{\lambda^{(j)}t}, \qquad j = 1, 2, \cdots, N.$$

Assume $v(t)$ is an irreducible stationary Markov chain, and let $\{\pi_k, k = 1, \cdots, N\}$ be its stationary initial distribution. Then, the extinction of the BPGEC is certain if and only if

$$\textstyle\sum_{k=1}^{N} \pi_k \lambda^{(k)} \leq 0.$$

Proof. We want to use Theorem A, but this theorem applies when the environmental process is a discrete parameter Markov chain. Thus, we have to look at our process not "continuously" in time but "discretely". Hence, the idea is to "observe" the process only at the jump times of the chain $v(t)$.

Let us recall that τ_i is the time of the i-th jump of $v(t)$, $i = 1, 2, \cdots$, and let $\tau_0 \equiv 0$. Let T_i be the waiting time of $v(t)$ in state $v(\tau_i)$, i.e., $T_i = \tau_{i+1} - \tau_i$, $i = 0, 1, 2, \cdots$.

Define a discrete time Markov chain $\{\eta_i : i = 0, 1, 2, \cdots\}$ with state space $\Theta = \{1, 2, \cdots, N\} \times \boldsymbol{R}$, equipped with the σ-algebra

$$\mathcal{B} = \{(A, B) : A \subset \{1, 2, \cdots, N\}, \text{ and } B \text{ is a Borel set in } \boldsymbol{R}\},$$

by

$$\eta_i = (v(\tau_i), T_i), \qquad i = 0, 1, 2, \cdots.$$

Now, associate to each η_i a p.g.f. by means of the mapping

$$\eta_i \rightarrow f_{\eta_i}(s) = F^{(v(\tau_i))}(s, T_i).$$

Next, consider the model of Athreya and Karlin corresponding to the Markov chain $\{\eta_i, i = 0, 1, 2, \cdots\}$ and to the family of p.g.f. $\{f_{\eta_i}, i = 0, 1, 2, \cdots\}$.

Clearly, the extinction of this branching process in a random environment is certain if and only if the extinction for our BPGEC is certain.

Let $\tilde{P} : \mathcal{B} \rightarrow \boldsymbol{R}$ be the stationary initial distribution of the chain $\{\eta_i, i = 0, 1, 2, \cdots\}$. By Theorem A, the extinction is certain if and only if

$$\textstyle\int_\Theta \operatorname{Log}\,(f_\theta'(1))\, d\tilde{P}(\theta) \leq 0,$$

i.e., if and only if, writing (j, t) instead of θ,

$$\int_\Theta \operatorname{Log}\left(\frac{\partial}{\partial s} F^{(j)}(s, t)\,|_{s=1}\right) d\tilde{P}(j, t) \leq 0,$$

or equivalently, if and only if

$$\textstyle\int_\Theta \operatorname{Log}\,(e^{\lambda^{(j)}t})\, d\tilde{P}(j, t) = \int_\Theta \lambda^{(j)} t\, d\tilde{P}(j, t) \leq 0.$$

To obtain $\tilde{P}$, we proceed as follows: for $j \in \{1, 2, \cdots, N\}$ and B a Borel set in $\boldsymbol{R}$, define

$$R_n(j, B) = \frac{1}{n} \textstyle\sum_{k=0}^{n} I_{\{j\}}(v(\tau_k)) I_B(T_k)$$

where I_A denotes the indicator function of the set A. $R_n(j, B)$ is the average number of times the chain $\{\eta_i, i = 0, 1, 2, \cdots\}$ visits the set (j, B) up to time τ_n. Let $0 \leq \tau_{i_1} < \tau_{i_2} < \cdots < \tau_{i_{n(j)}} \leq \tau_n$ be the times when $v(t)$ visits the state j within its first n jumps. Then

$$R_n(j, B) = \frac{1}{n} \sum_{k=1}^{n(j)} I_B(T_{i_k}) = (n(j)/n) \sum_{k=1}^{n(j)} I_B(T_{i_k})/n(j).$$

Now, let $t \to \infty$. Since in a finite state space irreducible Markov chain all states are positive recurrent (cf. Breiman (1969), Cor. 6.31), then we have $n(j)/n \to \tilde{\pi}(j)$, where $\{\tilde{\pi}(j), j = 1, 2, \cdots, N\}$ is the stationary initial distribution of the first jump transition probabilities of $v(t)$ (cf. Breiman (1969), p. 213). From this and the strong law of large numbers, we obtain $R_n(j, B) \to \tilde{\pi}(j)P[W(j) \in B]$ a.s. as $n \to \infty$, where $W(j)$ denotes the waiting time of $v(t)$ in state j. Therefore we claim that $\tilde{P}(j, B) = \tilde{\pi}(j)P[W(j) \in B]$.

To prove that this is correct, we have to show that

$$\tilde{P}(j, B) = \int_\Theta P[(j, B) \mid (k, t)]\, d\tilde{P}(k, t),$$

where

$$\{P[(j, B) \mid (k, t)], 1 \leq j \leq N, 1 \leq k \leq N, t \geq 0, \text{ and } B \text{ a Borel set in } \boldsymbol{R}\}$$

are the transition probabilities of $\{\eta_i, i = 0, 1, 2, \cdots\}$. From the definition of η_i, we have

$$P[(j, B) \mid (k, t)] = p(j \mid k)P[W(j) \in B],$$

where

$$\{p(j \mid k), j = 1, 2, \cdots, N, k = 1, 2, \cdots, N, j \neq k\}$$

are the first jump transition probabilities of $v(t)$.

Therefore

$$\begin{aligned} \int_\Theta P[(j, B) \mid (k, t)]\, d\tilde{P}(k, t) &= \sum_{k=1}^N \int_{\boldsymbol{R}} p(j \mid k)P[W(j) \in B]\tilde{\pi}(k)P[W(k) \in dt] \\ &= P[W(j) \in B] \sum_{k=1}^N p(j \mid k)\tilde{\pi}(k) \int_{\boldsymbol{R}} P[W(k) \in dt] \\ &= P[W(j) \in B]\tilde{\pi}(j) = \tilde{P}(j, B) \end{aligned}$$

which proves the claim.

Hence, going back to the problem of extinction we have

$$\begin{aligned} \int_\Theta \lambda^{(j)} t\, d\tilde{P}(j, t) &= \sum_{j=1}^N \tilde{\pi}(j)\lambda^{(j)} \int_{\boldsymbol{R}} tP[W(j) \in dt] \\ &= \sum_{j=1}^N \tilde{\pi}(j)\lambda^{(j)} E[W(j)] = \sum_{j=1}^N \lambda^{(j)} \tilde{\pi}(j)/q_j, \end{aligned}$$

but $\tilde{\pi}(j)/q_j = c\pi_j$, where $c > 0$ is the appropriate normalizing constant. Thus

$$\int_\Theta \lambda^{(j)} t\, d\tilde{P}(j, t) = \sum_{j=1}^N c\pi_j \lambda^{(j)}.$$

Hence, the extinction is certain if and only if $\sum_{j=1}^N \pi_j \lambda^{(j)} \leq 0$. Q.E.D.

3. Expected number of particles at time t

Let us recall that $G_i^{(j)}(s, t)$ denotes the p.g.f. of $Z(t)$ under the condition $v(0) = j$ and $Z(0) = i$, and $G_1^{(j)}(s, t)$, which is simply written as $G^{(j)}(s, t)$, satisfies

$$(3.1) \quad G^{(j)}(s, t) = F^{(j)}(s, t)e^{q_{jj}t} + \sum_{k\neq j} \int_0^t F^{(j)}(G^{(k)}(s, t - r), r)q_{jk}e^{q_{jj}r}\, dr.$$

Let $M_j(t)$ denote the expected number of particles at time t, if the process started with one particle in environment j, i.e.,

$$M_j(t) = \frac{\partial}{\partial s} G^{(j)}(s, t)\,|_{s=1}.$$

Before obtaining $M_j(t)$ from equation (3.1), note, by using the chain rule, that

$$\frac{\partial}{\partial s} F^{(j)}(G^{(k)}(s, t - r), r)\,|_{s=1} = e^{\lambda(j)r}M_k(t - r),$$

where we have used $G^{(k)}(1, t - r) = 1$ and $\partial/\partial s\, F^{(j)}(s, r)\,|_{s=1} = e^{\lambda(j)r}$ (see Theorem 2.2). Therefore, by differentiating both sides of (3.1), we obtain

$$M_j(t) = e^{(\lambda(j)+q_{jj})t} + \sum_{k\neq j} \int_0^t M_k(t - r)q_{jk}e^{(\lambda(j)+q_{jj})r}\, dr,$$

or, by a change of variable,

$$M^j(t) = e^{(\lambda(j)+q_{jj})t} + \sum_{k\neq j} q_{jk}e^{(\lambda(j)+q_{jj})t} \int_0^t e^{-(\lambda(j)+q_{jj})r}M_k(r)\, dr.$$

Hence, $M_j(t)$ satisfies

$$(3.2) \quad \begin{aligned} &\frac{d}{dt} M_j(t) = M_j(t)(\lambda^{(j)} + q_{jj}) + \sum_{k\neq j} q_{jk} M_k(t) \\ &M_j(0) = 1, \qquad j = 1, 2, \cdots, N. \end{aligned}$$

Let

$$(3.3) \quad \tilde{M}(t) = \begin{pmatrix} M_1(t) \\ M_2(t) \\ \vdots \\ M_N(t) \end{pmatrix},$$

$$(3.4) \quad A = \begin{bmatrix} \lambda^{(1)} + q_{11} & q_{12} & \cdots & q_{1N} \\ q_{21} & \lambda^{(2)} + q_{22} & & \vdots \\ \vdots & & & \vdots \\ q_{N1} & q_{N2} & \cdots & \lambda^{(N)} + q_{NN} \end{bmatrix}$$

i.e., $A = Q + \Lambda$, where $\Lambda = \operatorname{diag}(\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(N)})$.
Then, we can write (3.2) as

$$\frac{d}{dt} \tilde{M}(t) = A\tilde{M}(t), \qquad \tilde{M}(0) = \tilde{1},$$

where

$$\tilde{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

The solution of this differential equation is

$$\tilde{M}(t) = e^{At}(\tilde{1}),$$

where $\{e^{At}, t \geq 0\}$ is the semigroup generated by A.

Summarizing this section, we have

THEOREM 3.1. *Let $M_j(t)$ be the expected number of particles at time t given that $v(0) = j$ and $Z(0) = 1, j = 1, 2, \cdots, N$. Let $\tilde{M}(t)$ and A be defined by* (3.3) *and* (3.4) *respectively, then*

$$\tilde{M}(t) = e^{At}(\tilde{1})$$

where $\tilde{1}$ is the column vector with all its entries equal to 1, *and $\{e^{At}, t \geq 0\}$ is the semigroup generated by A.*

4. Limit theorems

In this section two limit theorems concerning our model are obtained. The first one results by a direct application of a known theorem about random evolutions and, roughly speaking, says that if the time scale is "speeded up", i.e., if the Markov chain jumps faster and faster, then the corresponding BPGEC behaves more and more as a single branching process in an environment which is the "weighted average" of the different environments. The weights are placed according to the stationary initial distribution of the Markov chain. The second theorem gives the rate of growth of $Z(t)$ and it is obtained by constructing from our process a nonnegative martingale and applying the martingale convergence theorem.

For ease of reference, we state without proving the following theorem.

THEOREM B (Kurtz (1972), Theorem 2.1). *Let $\{X(t), t \geq 0\}$ be a pure jump process with state space S. Suppose S is a separable, locally compact metric space, and there is a measure μ on the Borel subsets of S such that $\mu(S) = 1$, and*

$$P\left[\lim_{t\to\infty} \frac{1}{t} \int_0^t g(X(u)\, du) = \int_S g(x)\mu\,(dx)\right] = 1$$

for every real, bounded, continuous function g on $[0, \infty)$. For each $x \in S$, let $\{T_x(t), t \geq 0\}$ be a semigroup of linear operators on a Banach space L, with infinitesimal operator A_x, which satisfies $\| T_x(t) \| \leq e^{\alpha t}$ for some α independent of x. Let D be the set of $f \in L$ such that $A_x f\colon S \to L$ is a bounded continuous function of x, and define A on D by

$$Af = \int_S A_x f\mu\,(dx).$$

If D is dense in L, and the range of $\mu - A$ is also dense in L for some $\mu > \alpha$, then the closure of A is the infinitesimal operator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ defined on L, and

$$P[\lim_{\lambda\to\infty} T_\lambda(t)f = T(t)f] = 1 \textit{ for every } f \in L,$$

where

$$T_\lambda(t) = T_{\xi_0}\left(\frac{1}{\lambda}\Delta_0\right) T_{\xi_1}\left(\frac{1}{\lambda}\Delta_1\right) \cdots T_{\xi_{N(\lambda t)}}\left(\frac{1}{\lambda}\Delta_{\lambda t}\right),$$

$\xi_0, \xi_1, \xi_2, \cdots$ is the succession of states visited by $X(t)$, and $\Delta_0, \Delta_1, \Delta_2, \cdots$ are the respective sojourn times in each state, $N(t)$ is the number of transitions of $X(t)$ before time t, and $\Delta_t = t - \sum_{k=0}^{N(t)-1} \Delta_k$.

We now state and prove the aforementioned theorems about our model.

THEOREM 5.1. *If the Markov chain $v(t)$ is irreducible and stationary, with $\{\pi_j\}_{j=1}^N$ as its stationary initial distribution, then $A = \sum_{j=1}^N \pi_j C^{(j)}$ is the infinitesimal generator of a strongly continuous semigroup $\{S(t), t \geq 0\}$, defined on ℓ_∞, and*

$$\lim_{\epsilon\to 0} S^{(v(0))}(\epsilon\tau_1) S^{(v(\tau_1))}(\epsilon(\tau_2 - \tau_1)) \cdots S^{(v(\tau_{N(t/\epsilon)}))}(t - \epsilon\tau_{N(t/\epsilon)}) = S(t)$$

with probability 1.

Proof. This theorem follows immediately from Theorem B, just by noticing that the semigroups $\{S^{(j)}(t), t \geq 0\}, j = 1, 2, \cdots, N$, are contraction semigroups, and the stationary initial distribution $\{\pi_j\}_{j=1}^N$ of $v(t)$ satisfies, by the ergodic theorem,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t g(v(u))\, du = E_\pi[g(v(0))]$$

with probability 1. Here E_π denotes expectation with respect to the measure $\{\pi_j\}_{j=1}^N$. Q.E.D.

For the second limit theorem, let us define

$$W(t) = Z(t)/\exp\{\lambda^{(v(0))}(\tau_1) + \lambda^{(v(\tau_1))}(\tau_2 - \tau_1) + \cdots + \lambda^{(v(\tau_{N(t)}))}(t - \tau_{N(t)})\},$$

where $\lambda^{(j)}$ was defined in Theorem 2.2, let $\mathfrak{F}_t$ be the σ-algebra generated by $\{Z(s), 0 \leq s \leq t\}$ and $\{v(s), s \geq 0\}$, let $W_n = W(\tau_n)$, and $\mathfrak{F}_n = \mathfrak{F}_{\tau_n}$.

THEOREM 5.2. *$\{W_n, \mathfrak{F}_n\}_{n\geq 1}$ is a nonnegative martingale.*

Proof. All we have to show is the martingale property. Let $v(0) = j$, then we have to prove $E_j[W_n \mid \mathfrak{F}_{n-1}] = W_{n-1}$.

$$E_j[W_n \mid \mathfrak{F}_{n-1}]$$
$$= E_j[Z(\tau_n)/\exp\{\lambda^{(v(0))}(\tau_1) + \cdots + \lambda^{(v(\tau_{n-1}))}(\tau_n - \tau_{n-1})\} \mid \mathfrak{F}_{n-1}]$$

(by the branching property)

$$= E_j[Z(\tau_{n-1})X^{(v(\tau_{n-1}))}(\tau_n - \tau_{n-1})$$
$$\times (\exp\{\lambda^{(v(0))}(\tau_1) + \cdots + \lambda^{(v(\tau_{n-2}))}(\tau_{n-1} - \tau_{n-2}) + \lambda^{(v(\tau_{n-1}))}(\tau_n - \tau_{n-1})\})^{-1}|\mathfrak{F}_{n-1}]$$
$$= W_{n-1}E_j[X^{(v(\tau_{n-1}))}(\tau_n - \tau_{n-1})$$
$$\times (\exp\{\lambda^{(v(\tau_{n-1}))}(\tau_n - \tau_{n-1})\})^{-1} \mid \mathfrak{F}_{n-1}] = W_{n-1}. \quad \text{Q.E.D.}$$

By the martingale convergence theorem, we obtain from the theorem just proved

THEOREM 5.3. $\lim_{n\to\infty} W_n = W$ *exists with probability* 1.

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