A REMARK ON A CONVERSE OF TAYLOR'S THEOREM

BY J. J. RIVAUD

R. Abraham and J. Robbin in (1) proved essentially the following

THEOREM A. *Let X and Y be normed linear spaces, U an open subset of X,* $f_i: U \to L_s^k(X, Y)$ *for* $i = 0, 1, \cdots, r$ $(r \geq 0)$. *For any* $x \in U$ *and* $h \in X$ *such that* $x + h \in U$ *define*

$$\rho(x, h) = f_0(x + h) - \sum_{i=0}^{r} \frac{f_i(x)}{i!} h^i$$

Suppose that:

(i) *Each* f_i *is continuous* $(i = 0, 1, \cdots, r)$

(ii) $\| \rho(x, h) \| / \| h \|^r \to 0$ *as* $(x, h) \to (x_0, 0)$ $\qquad (x_0 \in U)$.

Then f_0 *is of class* C^r *and* $D^i f_0 = f_i$ $\qquad (i = 1, \cdots, r)$.

This result is an extension of the finite dimensional case proved by Glaeser (2). In this note, we prove that condition (ii) can be relaxed.

THEOREM B. *Let X and Y be normed linear spaces,* $U \subset X$ *an open subset,* $f_i: U \to L_s^k(X, Y)$ *for* $i = 0, 1, \cdots, r$ $(r \geq 0)$ *continuous maps and define* $\rho(x, h)$ *as in the above theorem.*

Suppose that for each $x \in U$

$$\frac{\rho(x, h)}{\| h \|^r} \to 0 \quad as \quad h \to 0.$$

Then

$$\frac{\rho(x, h)}{\| h \|^r} \to 0 \quad as \quad (x, h) \to (x_0, 0) \qquad (x_0 \in U)$$

The following lemmata are the steps of the proof.

LEMMA 1. *Let* $f: (a, b) \to \mathbf{R}$ *be a continuous map, denoted by*

$$D^+f(x) = \overline{\lim}_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

and

$$D_+f(x) = \underline{\lim}_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

If $D^+f(x) \geq 0$ *for all* $x \in (a, b)$, *then f is a non-decreasing monotone map. Analogously if* $D_+f(x) \leq 0$ *for all* $x \in (a, b)$ *then f is a non increasing monotone map.*

The proof can be found in [3] pp. 98.

LEMMA 2. *Let* $f: (a, b) \to \boldsymbol{R}$ *be a map of class* C^{r-1}. *Suppose that for every*

$x \in (a, b)$

$$\lim_{h\to 0} \frac{f(x+h) - \sum_{i=0}^{r-1} \frac{1}{i!} D^i f(x) h^i}{h^r} = 0$$

Then f is a polinomial of degree $\leq r - 1$.

Proof. We will prove that $D^+(D^{r-1}f)\ (x) \geq 0$ and $D_+(D^{r-1}f)\ (x) \leq 0$ for all $x \in (a, b)$, by lemma 1 $D^{r-1}f$ has to be constant and the lemma follows.

Since f is of class C^{r-1} we have

$$\rho(x, h) = f(x+h) - \sum_{i=0}^{r-1} \frac{1}{i!} D^i f(x) h^i$$

$$= \int_0^1 \frac{(1-t)^{r-2}}{(r-2)!} [D^{r-1}f(x+th) - D^{r-1}f(x)] h^{r-1}\, dt$$

Hence

$$\frac{\rho(x, h)}{h^r} = \int_0^1 \frac{(1-t)^{r-2} t}{(r-2)!} \left[\frac{D^{r-1}f(x+th) - D^{r-1}f(x)}{th}\right] dt$$

Suppose $D_+(D^{r-1}f)\ (x) \geq \epsilon > 0$ for some $x \in (a, b)$, then

$$\frac{D^{r-1}f(x+th) - D^{r-1}f(x)}{th} > \epsilon/2$$

for th small enough, hence

$$\frac{\rho(x, h)}{h^r} \geq \frac{\epsilon/2}{(r-2)!} \int_0^1 (1-t)^{r-2} \cdot t\, dt > 0$$

i.e.

$$\frac{\rho(x, h)}{h^r} \geq k > 0$$

for h small enough. This is a contradiction. Analogously if $D^+(D^{r-1}f)\ (x) \leq -\epsilon < 0$.

Corollary. *Let X be a normed linear space and* $f: (a, b) \to X$ *a* C^r *map such that for all* $x \in (a, b)$

$$\lim_{h\to 0} \frac{f(x+h) - \sum_{i=0}^{r-1} \frac{1}{i!} D^i f(x) h^i}{h^r} = 0$$

Then f is a polynomial of degree $\leq r - 1$.

The proof is obvious.

Lemma 3. *Let X be a normed linear space and*

$$f_i: (a, b) \to X \qquad i = 0, 1, \cdots, r$$

continuous maps such that for every $x \in (a, b)$

$$\lim_{h\to 0} \frac{f_0(x+h) - \sum_{i=0}^{r} \frac{1}{i!} f_i(x)h^i}{h^r} = 0$$

Then, f_0 *is of class* c^r *and* $D^i f_0 = f_i \ (i = 1, \cdots, r)$.

Proof (By induction on r). For $r = 1$ the result is trivial. Suppose it is valid for $r - 1$. It is clear that $f_0, \cdots, f_{r-1}$ meet the hypothesis of induction then we can suppose

$$f_i = D^i f_0 \qquad (i = 1, \cdots, r-1)$$

and

$$\lim_{h\to 0} \frac{f_0(x+h) - \sum_{i=0}^{r-1} \frac{1}{i!} D^i f_0(x)h^i - \frac{1}{r!} f_r(x)h^r}{h^r} = 0$$

Consider the map $g:(a, b) \to X$ defined by

$$g(x) = \int_{x_0}^{x} \int_{x_0}^{t_{r-1}} \cdots \int_{x_0}^{t_1} f_r\,(t_0)\, dt_0 \cdots dt_{r-1}$$

where $x_0 \in (a, b)$ is fixed.

The map g is of class C^r and $D^r g = f_r$. Let $m(x) = f_0(x) - g(x)$, then m is of class C^{r-1} and satisfies the condition of the corollary, hence it is a polynomial of degree $\leq r - 1$ and $D^r m = 0$ that implies the lemma.

Proof of theorem B. Let $x \in U$ and $h \in X$ with $\in h \| = 1$. It is enough to prove that

$$\rho(x, \lambda h) = \int_0^1 \frac{(1-t)^{r-1}}{(r-1)!} [f_r(x + t\lambda h) - f_r(x)](\lambda h)^r\, dt$$

because the continuity of f_r implies that

$$\frac{\rho(x, \lambda h)}{\lambda^r} \to 0 \quad \text{as} \quad x \to x_0 \quad \text{and} \quad \lambda \to 0.$$

Define $\varphi_i: (-\epsilon, \epsilon) \to X$ by

$$\varphi_i(\lambda) = f_i(x + \lambda h)h^i \qquad (i = 0, \cdots, r)$$

Since

$$\frac{\varphi_0(\lambda + \mu) - \sum_{i=0}^{r} \frac{1}{i!} \varphi_i(\lambda)\mu^i}{\mu^n} = \frac{f_0(x' + \mu h) - \sum_{i=0}^{r} \frac{1}{i!} f_i(x')(\mu h)^i}{\| \mu h \|^n}$$

where $x' = x + \lambda h$, we have that the conditions of the last lemma are fulfilled by the φ_i, then by the lemma and Taylor's theorem it follows that

$$\rho(x, \lambda h) = f(x + \lambda h) - \sum_{i=0}^{r} \frac{1}{i!} f_i(x)\ (\lambda h^i)$$

$$= \varphi_0(\lambda) - \sum_{i=0}^{r} \frac{1}{i!} \varphi_i(0)\lambda^i$$

$$= \int_0^1 \frac{(1-t)^{r-1}}{(r-1)!} [\varphi_r(t\lambda) - \varphi_r(0)]\lambda^r \, dt$$

$$= \int_0^1 \frac{(1-t)^{r-1}}{(r-1)!} [f_r(x + t\lambda h) - f_r(x)](\lambda h)^r \, dt.$$

Observe that a similar result for the remainders in the Whitney's extension theorem [4] is well known to be false.

In some places, Theorem A appears loosely stated (see [5]) apparently implying that in their proof they just need the weaker assumption on the remainder, which is not the case.

CENTRO DE INVESTIGACION DEL IPN, MEXICO, D. F.

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