A NOTE ON STIEFEL MANIFOLDS AND THE GENERALIZED J HOMOMORPHISM

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§1. Introduction

Let $V_{n,k}$ be the space of all k-frames in R^n, i.e., the space of real $k \times n$ matrices of rank k. For most purposes of homotopy theory, $V_{n,k}$ is interchangeable with $\mathbf{V}_{n,k}$, the classical Stiefel manifold of *orthonormal* k-frames in R^n. In fact, if τ denotes the involution which reverses the sign of a frame, then there is a τ-equivariant deformation retract from $V_{n,k}$ to $\mathbf{V}_{n,k}$.

Consider the map $f_{m,k}: S^{m-1} \to V_{m-1+k,k}$ defined by sending the vector $(x_1, \cdots, x_m) \in S^{m-1}$ to the following $k \times (m-1+k)$ matrix,

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdot\cdot & \cdots & x_m & & \\ & x_1 & x_2 & \cdot\cdot & \cdots & x_{m-1} & x_m & \\ & & \cdot\cdot & \cdot\cdot & \cdots & \cdots\cdot & \cdot & \\ & & & x_1 & x_2 & \cdots\cdot & \cdot & x_m \end{bmatrix}$$

where the undesignated entries are equal to zero. This defines a homotopy element $[f_{m,k}]$ in $\pi_{m-1}(V_{m-1+k,k})$. The purpose of the present note is to determine $[f_{m,k}]$ explicitly, in theorems 1 and 2 below.

Recall that $V_{m-1+k,k}$ is $(m$-$2)$-connected, with $\pi_{m-1}(V_{m-1+k,k})$ cyclically generated by an element e_{m-1}, which has infinite order if $k = 1$ or if m is odd, and has order 2 otherwise. We shall prove

Theorem 1. *Let m be even, then* $[f_{m,k}] = \binom{m-1+k}{m} e_{m-1}$.

Theorem 2. *Let $m = 2p+1$ be odd. Then $[f_{m,k}] = 0$ if k is even, and $[f_{m,k}] = \binom{p+l}{p} e_{m-1}$ if $k = 2l+1$ is odd.*

The proof of theorem 2 depends on the use of the *generalized* J *homomorphism* which we now explain. Given a map g: $S^{q-1} \to V_{n,k}$ representing an element $[g]$ in $\pi_{q-1}(V_{n,k})$, one can form the *adjoint map*

$$g^{\#}: S^{q-1} \times R^k \to R^n$$

defined by $g^{\#}(x, y) = y \cdot g(x)$, where $y \in R^k$ is thought of as a $1 \times k$ matrix, and the dot denotes matrix multiplication. By radial extension on the one hand, and by normalization on the other, one can obviously use $g^{\#}$ to define two closely related maps $g^{\#}{}_1: R^q \times R^k \to R^n$ and $g^{\#}{}_2: S^{q-1} \times S^{k-1} \to S^{n-1}$ respectively, also referred to as the *adjoint map* of g. The Hopf construction applied to $g^{\#}{}_2$ gives a map $S^{q+k-1} \to S^n$, which we shall denote interchangeably by $J(g)$, $J(g^{\#}{}_1)$ or $J(g^{\#}{}_2)$. The generalized J homomorphism is then the map $J: \pi_{q-1}(V_{n,k}) \to \pi_{q+k-1}(S^n)$ given by $[g] \to J(g)$. Its properties are summarized in §2. Of course, when $k = n$, $V_{n,k}$ is the general linear group, and J simply reduces to the classical J homomorphism.

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Let $\phi_{m,k}:R^m \times R^k \to R^{m+k-1}$ be the nonsingular bilinear map [2] given by "polynomial multiplication;" that is, for $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_k)$, $\phi_{m,k}(x, y) = (z_1, \cdots, z_{m+k-1})$ where the z_j's are defined such that $(\Sigma y_j t^j)(\Sigma x_j t^j) = \Sigma z_j t^j$ as polynomials in t. Then it is easy to verify that $\phi_{m,k}$ is the adjoint of $f_{m,k}$, and our explicit determination of $[f_{m,k}]$ can be reinterpreted as

COROLLARY 3. *The map $J(\phi_{m,k}):S^{m+k-1} \to S^{m+k-1}$ has degree $2\binom{p+\ell}{p}$ if $m = 2p + 1$ and $k = 2\ell + 1$, and has degree zero otherwise.*

Thus, one can say that self maps of degree $\pm 2\binom{n}{r}$ of an odd dimensional sphere S^{2n-1} can "arise from nonsingular bilinear maps", via the Hopf construction process explained above.

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§2. The generalized J homomorphism

We begin by observing that there is a canonical map

$$\theta: V_{n,k} \times S^{k-1} \to S^{n-1}$$

defined by $\theta(A, y) = y \cdot A / \| y \cdot A \|$, where $A \in V_{n,k}$ is a $k \times n$ matrix. The Hopf construction applied to θ gives $H_\theta: V_{n,k} * S^{k-1} \to S^n$. If now $g:S^{q-1} \to V_{n,k}$ is any map, then it is easy to verify that $J(g)$ is equal to the following composite

$$S^{k+q-1} = S^{q-1} * S^{k-1} \xrightarrow{g*1} V_{n,k} * S^{k-1} \xrightarrow{H_\theta} S^n.$$

Consequently we have

PROPOSITION (2.1). *The map $J:\pi_{q-1}(V_{n,k}) \to \pi_{q+k-1}(S^n)$ factors as*

$$\pi_{q-1}(V_{n,k}) \xrightarrow{\Sigma^k} \pi_{q+k-1}(\Sigma^k V_{n,k}) \xrightarrow{H_{\theta *}} \pi_{q+k-1}(S^n),$$

where Σ^k is the k-fold suspension. In particular, J is indeed a homomorphism.

COROLLARY (2.2). *Suppose $k > 1$ and $n - k$ is odd. Then $J:\pi_{n-k}(V_{n,k}) \to \pi_n(S^n)$ must be the zero homomorphism.*

For in this case, J is a homomorphism $Z_2 \to Z$.

PROPOSITION (2.3). *For any maps $g:S^{q-1} \to V_{n,k}$ and $h:s^r \to S^{q-1}$, one has the formula*

$$J(g \circ h) = J(g) \circ (\Sigma^k h) \in \pi_{r+k}(S^n).$$

The proof is a simple application of proposition (2.1).

We next remark that if $g:S^{q-1} \to V_{k,n}$ has adjoint map $g^{\#}{}_1:R^q \times R^k \to R^n$, then $J(g)$ or $J(g^{\#}{}_1)$, when regarded as an element in $\pi_{q+k-1}(R^{n+1} - \{0\})$, has a convenient representative $S^{q+k-1} \to R^{n+1} - \{0\}$ given by

$$(\dagger) \qquad (x_1, \cdots, x_q, y_1, \cdots, y_k) \to (\Sigma x_j^2 - \Sigma y_j^2, 2g_1^{\#}(x, y)).$$

The proof is obtained by simply spelling out the definition of the Hopf construction (compare [3]). Using this remark, one can evaluate J on the generator e_{n-k} of $\pi_{n-k}(V_{n,k})$:

PROPOSITION (2.4). *$J(e_{n-k})$ in $\pi_n(S^n) = Z$ is equal to zero if $n-k$ is odd, and is equal to twice a generator if $n-k$ is even.*

Proof. It is well known that a representative of e_{n-k} factors as $S^{n-k} \xrightarrow{e} V_{n-k+1,1} \xrightarrow{i} V_{n,k}$, where $e(x_1 \cdots x_{n-k+1})$ = the one-rowed matrix $[x_1, \cdots, x_{n-k+1}]$, and i is the natural inclusion

$$[x_1, x_2, \cdots, x_{n-k+1}] \to \left[\begin{matrix} x_1 & x_2 & \cdots & x_{n-k+1} & 0 \cdots 0 \\ 0 & 0 & \cdots & 0 & 1 \cdots 0 \\ & & \cdots & & \cdots \\ 0 & 0 & \cdots & 0 & 0 \cdots 1 \end{matrix}\right] \Bigg\} k \text{ rows}.$$

Using the explicit formula ($\dagger$), it is a simple homotopy exercise to deduce that $J(e_{n-k})$ is the (k-1)-fold suspension of $J(e) \in \pi_{n-k+1}(S^{n-k+1})$. Thus we are reduced to proving an appropriate assertion for $J(e)$, which corresponds to the special case $k = 1$ in the proposition.

When $k = 1$, $J(e_{n-1}) : S^n \to R^{n+1} - \{0\}$ is given by $(x_1, \cdots, x_n, y) \to (\Sigma x_j^2 - y^2, 2yx_1, \cdots, 2yx_n)$.

Notice that antipodal points have the same image, which forces the map to be inessential for even n. When n is odd, there is a standard homotopy trick, given in [3, lemma 4], to show that the map has degree two. This completes the proof.

COROLLARY (2.5). *Any stable homotopy class of spheres which is halvable lies in the image of an appropriate generalized J homomorphism.*

Proof. Let $u = 2v$ be the class in question. Choose $n - k$ even, and apply proposition (2.3) to the case where g equals e_{n-k}, and $h : S^r \to S^{n-k}$ is a representative of v.

It should of course be remarked that halvable classes are not the only ones to lie in the image of various generalized J homomorphisms. For instance, the image of the classical J homomorphism contains many nonhalvable homotopy classes of spheres.

§3. Proof of theorem 1

Let $PV_{n,k}$ denote the space obtained from $V_{n,k}$ by identifying each frame A with its opposite frame $-A$. Clearly, $PV_{n,k}$ has the same homotopy type of a "projective Stiefel manifold;" in particular, its cohomology is described in [1, theorem 1.6]. We shall require the following portion of that theorem: *when the* *binomial coefficient* $\binom{n}{n-k+1}$ *is even, the double covering map* π: $V_{n,k} \to PV_{n,k}$ *induces an epimorphism in (n − k) dimensional cohomology with mod 2 coefficients.*

To prove theorem 1, note that it is obviously true when $k = 1$. Thus we take m even, $k > 1$, so that $\pi_{m-1}(V_{m-1+k,k}) = Z_2$. It suffices then to show that $[f_{m,k}] = 0$ if and only if $\binom{m-1+k}{m}$ is even. To this end, observe that the map $f_{m,k}$ fits into a commutative diagram

$$\begin{array}{ccc} S^{m-1} & \xrightarrow{f_{m,k}} & V_{m-1+k,k} \\ \downarrow \pi_0 & & \downarrow \pi \\ RP^{m-1} & \rightarrow & PV_{m-1+k,k}\,, \end{array}$$

where π_0 is the standard double cover. It follows from naturality that if $\binom{m-1+k}{m}$ is even, then

$$f^*_{m,k}: H^{m-1}(V_{m-1+k,k}\,; Z_2) \to H^{m-1}(S^{m-1}; Z_2)$$

is the zero homomorphism $Z_2 \to Z_2$, and this in turn implies that $[f_{m,k}] = 0$.

Suppose, conversely, that $[f_{m,k}] = 0$. Then it is possible to extend $f_{m,k}$ to a map $F_{m,k}: E_+^m \to V_{m-1+k,k}$ defined on the northern hemisphere E_+^m of S^m. Setting $F_{m,k}(x) = -F_{m,k}(-x)$ for all x in the southern hemisphere, we obtain an equivariant map $F_{m,k}: S^m \to V_{m-1+k,k}$, whose adjoint is a so-called *nonsingular skewlinear map* $F^{\#}_{m,k}: S^m \times R^k \to R^{m-1+k}$. But it is well-known [2, Corollary 2.2] that such an $F^{\#}_{m,k}$ enables one to construct k independent sections of $(m-1+k)\xi$ over RP^m, where ξ is the Hopf line bundle. Consequently, $(m-1+k)\xi$ has zero Stiefel-Whitney class in dimension m, which means that $\binom{m-1+k}{m}$ must be even. This establishes theorem 1. As a matter of fact, we have shown that for m even and $k > 1$, any equivariant map from S^{m-1} to $V_{m-1+k,k}$ represents $\binom{m-1+k}{k}e_{m-1}$.

§4. Proof of theorem 2

In case $m = 2p + 1$ is odd, $\pi_{m-1}(V_{m-1+k,k}) = Z$, and there is a unique integer d such that $[f_{m,k}] = de_{m-1}$. It follows from proposition (2.4) that $J(f_{m,k})$ has degree $2d$. If k is even, we take advantage of the fact that both $f_{m,k}$ and $f_{k,m}$ have polynomial multiplication maps ($\phi_{m,k}$ and $\phi_{k,m}$ of §1) as their adjoints, and conclude that

$$J(f_{m,k}) = J(\phi_{m,k}) = \pm\, J(\phi_{k,m}) = \pm\, J(f_{k,m}) = 0,$$

by corollary (2.2). This gives $d = 0 = [f_{m,k}]$.

It remains now to settle the case when $k = 2\ell + 1$ is odd. The argument hinges on the following key lemma, which corresponds to the special case $m = 3$.

LEMMA (4.1). *Let $k = 2\ell + 1$ be odd. Then $f_{3,k} : S^2 \to V_{k+2,k}$ represents $(\ell + 1)$ times the generator in $\pi_2(V_{k+2,2})$.*

Proof. Abbreviate $f_{3,k}$ by f. For $x = (x_1, x_2, x_3)$ in S^2, $f(x)$ is the $k \times (k + 2)$ matrix given by

$$f(x) = \begin{bmatrix} x_1 & x_2 & x_3 & & & & \\ & x_1 & x_2 & x_3 & & & \\ & & x_1 & x_2 & x_3 & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & x_1 & x_2 & x_3 \end{bmatrix}$$

For $0 \leq t \leq 1$, let $f_t(x)$ be the matrix obtained from $f(x)$ by

(i) replacing the zero in entries $(2, 1), (4, 3), (6, 5), \cdots, (2\ell, 2\ell - 1)$ by $-tx_2$, and

(ii) replacing the x_2 in entries $(2, 3), (4, 5), \cdots, (2\ell, 2\ell + 1)$ by $(1 - t)x_2$.

This is a homotopy of f in $V_{m-1+k,k}$, ending in a map f_1 given by

$$f_1(x) = \begin{bmatrix} X & X' & & & \\ & X & X' & & \\ & \cdot & \cdots & \cdot & \cdot \\ & & X & X' & 0 \\ & & & x_1 x_2 & x_3 \end{bmatrix}$$

where $X = \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}$, $X' = \begin{pmatrix} x_3 & 0 \\ 0 & x_3 \end{pmatrix}$ are 2×2 blocks. For $1 \leq t \leq 2$, let $f_t(x)$ be obtained from $f_1(x)$ by replacing each X' with

$$\begin{bmatrix} (2 - t)x_3 & t - 1 \\ -(t - 1) & (2 - t)x_3 \end{bmatrix}.$$

This is a further homotopy of f_1 into a map f_2, $f_2(x)$ being the result of substituting each X' in $f_1(x)$ by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Using a path in $S0(2)$ to join $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we can eventually homotop f to a map f_3, with

$$f_3(x) = \begin{bmatrix} X & I_2 & & & \\ & X & I_2 & & \\ & & \cdot\cdot & \cdot & \\ & & X & I_2 & 0 \\ & & & x_1 x_2 & x_3 \end{bmatrix}.$$

By standard use of elementary column and row operations (compare [2, §4]), it is possible to further deform f_3 to f_4, where

$$f_4(x) = \begin{bmatrix} \rho(x_1, x_2) & x_3 & 0 & \cdots & 0 \\ 0 & 0 & & I_{2\ell} & \end{bmatrix},$$

in which $\rho(x_1, x_2) = (-1)^{\ell}(x_1, x_2)\begin{pmatrix} x_1, & x_2 \\ -x_2 & x_1 \end{pmatrix}^{\ell}$ is a two dimensional vector. If we think of ρ as a self map of the circle, then its degree is $\ell + 1$. Hence our original map $f = f_{3,k}$ represents $(\ell + 1)$ times the generator, as is to be proved.

COROLLARY (4.2). *Let $\phi_{3,2\ell+1}$ be the polynomial multiplication map $R^3 \times R^{2\ell+1} \to R^{2\ell+3}$. Then $J(\phi_{3,2\ell+1})$ has degree* $2(\ell + 1)$.

For the balance of this section, we adopt the notation that if W is a vector space, then W_* denotes W with the origin deleted.

COROLLARY (4.3). *The restriction map $\phi^*_{3,2\ell+1} : R_*^3 \times R_*^{2\ell+1} \to R_*^{2\ell+3}$ of $\phi_{3,2\ell+1}$ has homological degree $2(\ell + 1)$.*

To prove theorem 2, introduce a map

$$\psi_p : \underbrace{R^3 \times \cdots \times R^3}_{p} \longrightarrow R^{2p+1}$$

defined via polynomial multiplications, in the obvious way. The restriction $\psi^*_p : R_*^3 \times \cdots \times R_*^3 \to R_*^{2p+1}$ of ψ_p has homological degree $2^{p-1}p!$, as follows from corollary (4.3) by an easy induction. Consider now the commutative diagram

$$\begin{array}{ccc} (R_*^3)^p \times (R_*^3)^{\ell} & & \\ \Big\downarrow \psi^*_p \times \psi^*_{\ell} & \searrow^{\psi^*_{p+\ell}} & \\ R_*^{2p+1} \times R_*^{2\ell+1} & \xrightarrow[\phi^*_{2p+1,2\ell+1}]{} & R_*^{2p+2\ell+1}. \end{array}$$

It follows easily by functoriality that $\phi^*_{2p+1,2\ell+1}$ has homological degree $2\binom{p+\ell}{p}$. Since this is also the degree of $J(\phi_{2p+1,2\ell+1})$, proposition (2.4) implies that $[f_{2p+1,2\ell+1}]$ is equal to $\binom{p+\ell}{p}$ times the generator, which is what theorem 2 asserts.

Corollary 3 of §1 has already been established in the course of the above proof.

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