SYMMETRIC AXIAL MAPS AND EMBEDDINGS OF PROJECTIVE SPACES

By A. J. Berrick, S. Feder and S. Gitler

1. Introduction

We consider smooth embeddings of real projective n-space P^n in Euclidean $(n + k) -$ space R^{n+k}, and their relation to *symmetric axial maps* (SAMs) of type (n,k), that is, to maps $P^n \times P^n \to P^{n+k}$ which are both axial (i.e. lift to $S^n \times S^n \to S^{n+k}$ but to neither $S^n \times P^n \to S^{n+k}$ nor $P^n \times S^n \to S^{n+k}$) and symmetric (i.e. equivariant with respect to the twisting involution on $P^n \times P^n$ and trivial involution on P^{n+k}). Note that a homotopy between SAMs must be a homotopy through axial maps but not necessarily through symmetric axial maps; the latter condition defines a *symmetric homotopy.* This note shows that homotopy and symmetric homotopy classes of SAMs correspond to regular homotopy and isotopy classes of embeddings respectively. More precisely, the following theorems are established.

THEOREM 1.1. a) *Each isotopy class of embeddings* $P^n \subset R^{n+k}$ *gives rise to a unique symmetric homotopy class of SAMs of type (n,k).*

b) *Each regular homotopy class of embeddings* $P^n \subset R^{n+k}$ *gives rise to a unique homotopy class of SAMs of type (n,k).*

THEOREM 1.2. *Suppose* $2k \geq (n + 1)$.

a) *Each symmetric homotopy class of SAMs of type (n,k) gives rise to an isotopy class of embeddings* $P^n \subset R^{n+k+1}$*; this assignment is* $1 - 1$ *if* $2k \geq (n + 2)$.

b) *Each homotopy class of SAMs of type (n,k) gives rise to a regular homotopy class of embeddings* $P^n \subset R^{n+k+1}$*; this assignment is* $1 - 1$.

Remarks (i) In view of [1] (2.1), the numerical condition of Theorem 1.2 may be dropped, save when $n \leq 15$; for these cases it is an open question whether the condition is required. (ii) When SAMs have the additional property that their covering map $S^n \times S^n \to S^{n+k}$ is *bilinear*, then by [6] they induce embeddings $P^n \subset R^{n+k}$. On the other hand, there exists examples [2] of merely homotopy symmetric axial maps (= *symmaxial maps*) of type (n,k) where P^n does not embed in R^{n+k}. It is not known whether there exists a SAM of type (n,k) for some n,k such that P^n does not embed in R^{n+k}.

In §2 we show that the study of regular homotopy and isotopy classes of embeddings may be replaced by consideration of suitable equivariant homotopy classes of maps from a Stiefel manifold to a sphere. Proofs of the theorems ensue in §3.

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2. Embeddings and Stiefel manifolds

Let D denote the dihedral group of order 8, with subgroup H the elementary Abelian group of order 4. H and D act on the $(n + k - 1)$-sphere S^{n+k-1} and on $V_n = V_{n+1,2}$ (the Stiefel manifold of ordered pairs of orthonormal vectors in R^{n+1}), as follows. D acts faithfully on V_n by sending the pair $(x,y) \in V_n$ to the pairs $(\pm x, \pm y)$, $(\pm y, \pm x)$, while H maps (x,y) to $(\pm x, \pm y)$. Then the generators of D corresponding to $(x,y) \to (y, -x)$ and $(x,y) \to (-x,y)$ act antipodally on S^{n+k-1}.

PROPOSITION 2.1. a) *There is a function from the set of isotopy classes of embeddings $P^n \subset R^{n+k}$ to the set of D-equivariant homotopy classes of D-equivariant maps from V_n to S^{n+k-1}, which is surjective if $2k \geq (n + 3)$ and bijective if $2k \geq (n + 4)$.*

b) *There is a function from the set of regular homotopy classes of embeddings $P^n \subset R^{n+k}$ to the set of H-equivariant homotopy classes of D-equivariant maps from V_n to S^{n+k-1}, which is injective if $2k \geq (n + 2)$ and bijective if $2k \geq (n + 3)$.*

c) *There is a function from the set of regular homotopy classes of immersions $P^n \subseteq R^{n+k}$ to the set of H-equivariant classes of H-equivariant maps from V_n to S^{n+k-1}, which is surjective if $2k \geq (n + 1)$ and bijective if $2k \geq (n + 2)$.*

Proof a) Haefliger [3] described the function from the set of isotopy classes as far as to the set of Z_2 − homotopy classes of Z_2 − maps from $P^n \times P^n - \Delta$ to S^{n+k-1}. Equivalently, in view of [5] (2.6), its range is a set of D-homotopy classes of D-maps from V_n to S^{n+k-1}, a set equivalent to ours via the D-involution Ψ on V_n given by $\Psi(x,y) = ((x + y)/\sqrt{2},(x - y)/\sqrt{2})$.

b) This is a consequence of a) and c), together with [4] (2.2). Note that the construction Φ of [4] §2 corresponds here to composition with Ψ.

c) Since V_n/H is just the projective tangent bundle of P^n, c) is [4] (4.2).

3. Proofs of theorems

Proof of (1.1). In view of (2.1) it suffices to associate to each D-homotopy (resp. H-homotopy) class to D-maps from V_n *to* S^{n+k-1} a symmetric homotopy (resp. homotopy) class of SAMs of type (n,k).

Let ξ be the real line bundle over V_n/D comprising triples $[x,y,t]$, $(x,y) \in V_n$, $t \in R$, with $[x,y,t] = [y, -x, -t] = [-x, y, -t]$; let η be the canonical line bundle over P^{n+k-1}, comprising pairs $[z,t]$, $z \in S^{n+k-1}$, $t \in R$, with $[z,t] = [-z, -t]$. Then a D-map $f: V_n \to S^{n+k-1}$ induces a bundle map $\bar{f}: \xi \to \eta$ given by $\bar{f}([x,y,t]) = [f(x,y),t]$. Likewise a D-homotopy induces a linear homotopy of bundle maps. However, if $\pi: V_n/H \to V_n/D$ is the double covering projection, then an H-homotopy induces a linear homotopy of bundle maps only from $\pi^*\xi$ to η.

Now observe the following relative homeomorphisms of sphere and disc bundles. $(B\pi^*\xi, S\pi^*\xi) \to (P^n \times P^n, \Delta)$; $(B\xi,S\xi) \to ((P^n \times P^n)/Z_2,\Delta/Z_2)$ where the involution is that of interchanging the factors; $(B\eta,S\eta) \to (P^{n+k}, \pm e)$, $e = (0, \cdots, 0, 1) \in S^{n+k}$. The first two maps are given by: $[x,y,t] \to (\nu(x + ty),$

$\nu(y + tx))$ for $\nu:R^N - \{0\} \to S^{N-1}$, $w \to w/\|w\|$; and the last by $[z,t] \to \pm\, (te + (1 - t^2)^{1/2} i(z))$. Where i is the map from $S^{n+k-1} \to S^{n+k}$, defined by $(z_0, \cdots, z_{n+k}) \to (z_0, \cdots, z_{n+k}, 0)$.

It follows that the induced Thom space maps and homotopies $P^n \times P^n \to T\pi^*\xi \to T\eta$ are SAMs and symmetric homotopies (resp. homotopies), as required.

Remark. Let $g:P^n \to R^{n+k}$ be an embedding. Then it is not difficult to see that a representative of the symmetric homotopy class obtained from g by the above construction is covered by $h:S^n \times S^n \to S^{n+k}$, where

$$h(x,y) = \begin{cases} \nu((x\cdot y)e + i(\|x + y\|\cdot\|x - y\|[g(\pm\nu(x + y)) \\ \qquad - g(\pm\nu(x - y))])) \text{ if } x \neq \pm\, y, \\ (x\cdot y)e, \text{ if } x = \pm\, y. \end{cases}$$

Proof of (1.2). Translation from axial maps of type (n,k) to equivariant maps from V_n to S^{n+k} is given by restriction to $V_n \subset S^n \times S^n$ of the map $S^n \times S^n \to S^{n+k}$ covering the axial map. The remaining passage to embeddings of P^n in R^{n+k+1} is achieved by (2.1).

ST. JOHN'S COLLEGE, OXFORD
CENTRO DE INVESTIGACION DEL IPN. MEXICO, D. F.

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