SHRINKABILITY AND THE KC-PROPERTIES

BY A. GARCÍA-MÁYNEZ

1. Introduction

In [1] we introduced the concepts of *shrinkable space* and *KS-space* and proved, among other things, that every compact shrinkable *KC*-space is Hausdorff (corollary 2.6.2). In this paper we modify the definition of shrinkability so as to include non-compact spaces and introduce the notions of KC' and KC'' spaces, both weaker than the *KC* property (every compact set is closed). We proved that the class of KC' and *KS* spaces (which contains all regular and all Hausdorff spaces) is closed under arbitrary products and coincides with the class of *KR*-spaces (every compact subset is regular) and with the class of spaces whose square is KC'. Also, the product of a KC' and *KS*-space with a KC'-space is KC'. The class of KC' shrinkable spaces is much more restricted: a space X is KC' and shrinkable if and only if it is paracompact and locally compact (hence, this class is only invariant under finite products). From this last equivalence it follows easily that every KC' space which is the continuous image of a compact regular space is also regular.

2. Definitions

A space X is *shrinkable* if every open cover of X has a compact locally finite refinement. X is KC' if for every compact $A \subset X$ and every neighborhood U of A, we have $A^- \subset U$. X is KC'' if the closure of every compact subset of X is also compact. By abuse of the T_i-axioms, we shall say that a space X is T_1' if it has a closed network (i.e., for every $x \in X$ and every neighborhood U of x, there exists a closed set K such that $x \in K \subset U$) and X is T_2' if every two points of X with different closures have disjoint neighborhoods. Clearly, every Hausdorff and every regular space is T_2' and every T_2'-space is T_1'.

We also recall the following definitions:

A space X is *KC* (resp., *KS*, resp., *KR*) if every compact subset of X is closed (resp., shrinkable, resp., regular). X is *locally compact* if every $x \in X$ has a compact neighborhood (not necessarily as small as we please). Finally, X is *paracompact* if X is regular and every open cover of X has a locally finite refinement. (Warning: in this paper, a regular or a normal space need not be T_1).

3. Basic results

The following remarks can be easily obtained from the definitions. We omit their proofs.

(3.1) *A space X is T_i if and only if it is T_0 and T_i' ($i = 1, 2$).*

(3.2) *A space X is KC if and only if it is KC' and T_0.*

(3.3) *The properties KC′, T_1' and T_2' are all hereditary.*

(3.4) *Let A, B be compact subsets of a topological space X. Then A and B have disjoint neighborhoods if and only if $A \times B$ is disjoint from $\Delta(X)^-$ (the closure of the diagonal)*

3.4.1 COROLLARY. *In a T_2'-space X,* $(x, y) \in \Delta(X)^-$ *if and only if* $\{x\}^- = \{y\}^-$.

3.4.2 COROLLARY. *Every compact T_2'-space is regular. Hence, every T_2'-space is KR.*

(3.5) *Every T_2'-space is KC′ and every KC′-space is KC″ and T_1'. In particular, every regular space is KC′.*

(3.6) *A space X is KC′ if and only if every compact subset of X is KC′.*

(3.7) *If X is KC′, $A \subset X$ is compact and $x \in A^- - A$, then $A \cap \{x\}^- \neq \Phi$.*

4. Main results

We start this section with a theorem relating shrinkability to well known properties.

(4.1) *A space* X *is KC′ and shrinkable if and only if it is paracompact and locally compact.*

Proof (Sufficiency). Being regular, X is KC' (see 3.5), so we have to prove only that X is shrinkable. Let $\mathscr{U}$ be an open cover of X. For each $x \in X$, select $U_x \in \mathscr{U}$ such that $x \in U_x$. Let V_x be an open set with compact closure such that $x \in V_x \subset V_x^- \subset U_x$. Let $\mathscr{K}$ be a closed locally finite refinement of the cover $\{V_x \mid x \in X\}$. Clearly, $\mathscr{K}$ is a compact locally finite refinement of $\mathscr{U}$. (Necessity). We first prove X is regular. By hypothesis, every open cover of X has a compact locally finite refinement. Since X is also KC', every open cover of X has a closed locally finite refinement. This clearly implies X is normal. But every normal T_1'-space is regular. Hence, X is regular. Paracompactness follows now immediately. It remains to prove that every x_0 in X has a compact neighborhood. Let $\mathscr{H}$ be a compact locally finite cover of X. By 3.5, we can clearly assume that every element of $\mathscr{H}$ is also closed. Let V be an open set about x_0 intersecting, at most, finitely many elements of $\mathscr{H}$. Then V^- is compact because it is contained in a finite union of elements of $\mathscr{H}$.

4.1.1 COROLLARY. *If $\mathscr{V} = \{V_j \mid j \in J\}$ is an open cover of a KC′, shrinkable space* X, *then $\mathscr{V}$ has a precise locally finite refinement $\mathscr{H} = \{K_j \mid j \in J\}$ consisting of closed compact sets.*

The following theorem relates our present definition of shrinkability with the one given in [1]. It is an easy consequence of corollary 4.1.1.

(4.2) *A compact KC′ space X is shrinkable if and only if for each finite open cover $\{V_1, V_2, \cdots, V_n\}$ of X, there exists a compact cover $\{K_1, K_2, \cdots, K_n\}$ of X such that $K_i \subset V_i$ for each i.*

4.1 has the following important consequences:

(4.3) *A locally compact space X is regular if and only if it is KC′ and KS.* (Compare with corollary 2.6.3 in [1]).

Proof. The necessity follows from 4.1 and 3.6. To prove the sufficiency, we have only to check that every point $x \in X$ has a closed neighborhood which is regular as a subspace of X. But by 4.1, 3.3 and 3.5, the closure of any compact neighborhood of x is regular.

4.3.1 COROLLARY. *A space X is KC′ and KS if and only if it is KR.*

Proof. Follows from 4.3, 3.5 and 3.6

(4.4) *Let $f: X \to Y$ be continuous and onto. If X is KC′ and shrinkable and f maps closed sets onto compact sets, then Y is shrinkable.*

Proof. Since $Y = f(X)$, Y is compact and we have only to consider finite open covers. Let $\{V_1, V_2, \cdots, V_n\}$ be an open cover of Y. The cover $\{f^{-1}(V_1), \cdots, f^{-1}(V_n)\}$ of X has a closed refinement $\{K_1, K_2, \cdots, K_n\}$, where K_i is contained in $f^{-1}(V_i)$ for each i (see corollary 4.1.1). If $A_i = f(K_i)$, then A_i is compact, $Y = A_1 \cup \cdots \cup A_n$ and $A_i \subset V_i$ for each i. Hence, Y is shrinkable.

4.4.1 COROLLARY. *Let* f: X → Y *be continuous and onto. If X is compact and regular, then Y is shrinkable. Hence, if Y is KC′, then Y is also regular.*

We prove now two important product theorems:

(4.5) *The class of KC″ spaces is closed under arbitrary products.*

Proof. Let $X = \prod\{X_j \mid j \in J\}$, where each X_j is KC''. Let $C \subset X$ be compact and let $p_j: X \to X_j$ be the projection onto the j-th factor. Since $C \subset \prod\{p_j(C) \mid j \in J\}$ and each $p_j(C)^-$ is compact, we have, by Tychonoff's product theorem, that $\prod\{p_j(C)^- \mid j \in J\}$ is compact. Since this last set contains C^-, we deduce that C^- is also compact.

(4.6) *The class of KC′ and KS spaces is closed under arbitrary products.*

Proof. Let $X = \prod\{X_j \mid j \in J\}$, where each X_j is KC' and KS. By 4.3.1, it suffices to prove that every compact set C in X is regular. But this is immediate since each $p_j(C)$ is regular (by 4.1) and $C \subset \prod\{p_j(C) \mid j \in J\}$.

The following lemmas will be used to characterize the class of KC' and KS spaces.

(4.7) *Let X be compact. If $X^2 = X \times X$ is KC′, then X is regular.*

Proof. Let $x \in V$, V open. Since X is KC' (recall property KC' is hereditary), the set $K = \{x\}^-$ is compact and it is contained in V. Let $W = X - K$. Observe that the open set $T = (V \times V) \cup (W \times W)$ contains the diagonal $\Delta(X)$. Since X^2 is KC', we have $\Delta(X)^- \subset T$. By 3.4, the compact sets K, $X - V$ have disjoint neighborhoods. Hence, X is regular.

4.7.1 COROLLARY. *Let X be a space whose square X^2 is KC′. Then X is KC′ and KS.*

Proof. We have only to prove that every compact set L in X is regular (see 4.3.1). Since property KC' is hereditary, the set $L^2 = L \times L$ is KC'. By 4.7, L is regular and the proof is complete.

(4.8) *If X is KC' and Y is KC' and KS, the product $X \times Y$ is KC'.*

Proof. Assume first X and Y are both compact. Let $C \subset T \subset X \times Y$, where C is compact and T is open. Proceeding by contradiction, assume there exists a point (x, y) in $C^- - T$. For each $(a, b) \in K = C \cap (\{x\}^- \times Y)$, there exist disjoint open sets in $X \times Y$ containing (x, y) and (a, b). If not, for some $(x', y') \in K$, we have $(x, x') \in \Delta(X)^-$ and $(y, y') \in \Delta(Y)^-$. By 3.4.1, we have $\{y\}^- = \{y'\}^-$. Let R', S' be open sets in X, Y, respectively, such that $(x', y') \in R' \times S' \subset T$. Hence, since both factors are KC', $\{x'\}^- \times \{y'\}^- = \{x'\}^- \times \{y\}^- \subset R' \times S' \subset T$. But $\{x'\}^- \times \{y\}^- \subset \{x\}^- \times \{y\}^- \subset C^- - T$, a contradiction. We have then open sets U, V in X, Y, respectively, such that $(x, y) \in U \times V \subset U^- \times V^- \subset X \times Y - K$. Let p_1, p_2 be the projections of $X \times Y$ onto X and Y, respectively. Consider the compact set $A = p_1[((U \times V) \cap C)^- \cap C]$ in X. Observe $\{x\}^- \cap A = \Phi$, because $K \cap (U^- \times V^-) = \Phi$. On the other side, $x \in [p_1((U \times V) \cap C)]^- \subset A^-$, since (x, y) belongs to $((U \times V) \cap C)^-$ and p_1 is continuous. Hence, $x \in A^- - A$. But according to 3.7, $\{x\}^- \cap A \neq \Phi$, a contradiction.

To prove the general case, it is sufficient to prove that every compact set $C \subset X \times Y$ is KC' (see 3.6). But by the previous case, $p_1(C) \times p_2(C)$ is KC'. Since $C \subset p_1(C) \times p_2(C)$ and the property KC' is hereditary, we conclude that C is also KC'. This completes the proof.

We are ready now to prove the main result of this paper:

(4.9) *In a space X, the following properties are equivalent:*
1) *X is KC' and KS;*
2) *For every KC'-space Y, the product $X \times Y$ is KC';*
3) *X^2 is KC';*
4) *X is KR; and*
5) *Every power X^a is KR.*

Proof. The implication 1) $\Rightarrow$ 2) is precisely 4.8 (compare it with 3.2 in [2]). 2) $\Rightarrow$ 3) and 5) $\Rightarrow$ 1) are obvious. 3) $\Rightarrow$ 4) follows from 4.7.1 and 4.3.1. Finally, 4) $\Rightarrow$ 5) is a consequence of the product theorem 4.6.

We conclude this paper by remarking that 4.9 remains valid if we replace KC' by KC all the way through and replace the word "regular" in condition 4) by the word "Hausdorff".

CENTRO DE INVESTIGACIÓN DEL IPN, MEXICO, D.F.

REFERENCES

[1] A. GARCÍA-MÁYNEZ, *Some generalizations of the Hausdorff separation axiom*, Bol. Soc. Mat. Mexicana, **16,** 1 (1971), 38–41.

[2] A. GARCÍA-MÁYNEZ, *On KC-spaces*, Anales del Instituto de Matemáticas, UNIVERSIDAD NACIONAL AUTONOMA DE MEXICO. **15,** 1 (1975), 33–50.