ON APPROXIMATION NUMBERS, n-DIAMETERS AND MEASURE OF NON-COMPACTNESS

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1. Introduction

In [7] R. D. Nussbaum gave a characterization of the essential spectral radius $r_e(T)$ of F. Browder essential spectrum $\sigma_e(T)$, of a bounded linear operator T defined on a Banach space X. The basic tool used there was K. Kuratowski measure of non-compactness [5] and Goldenstein—Gohberg—Markus ball measure of non compactness [3]. The object of this note is to relate some of the concepts and results in [7] to the concepts of approximation numbers and n-diameters which are due to A. Pietsch [8] and A. N. Kolmogoroff [4] respectively (see propositions 6 and 7). Also we will give another characterization of the essential spectral radius $r_e(T)$ (Proposition 4), and relate this result to P. Enflo solution of the basis problem [2] in the following way: Let X be a Banach space and let $L(X)$ the Banach algebra of all bounded linear operators in X. Denote by $F(X)$ and $K(X)$ the two sided ideals in $B(X)$ consisting of all operators of finite rank and all compact operators, respectively. Then $K(X)$ is closed and $F(X) \subseteq K(X)$. A long outstanding open question has been to characterize the closure $A(X)$ of $F(X)$ in $L(X)$. In this direction I. Maddaus [6] proved that if X has a Schauder basis, then $A(X) = K(X)$. However, a recent result of P. Enflo [2] gives an example of a separable reflexive space X where $A(X) \neq K(X)$. Now, a consequence of Proposition 5 in this work is the following

THEOREM: *Let X be a Banach space and consider the Banach algebras $\tilde{X} = L(X)/K(X)$ and $\hat{X} = L(X)/A(X)$. Then for every $T \in L(X)$ we have: spectral radius of $(\tilde{T})$ = spectral radius of $(\hat{T}) = r_e(T)$, where $\tilde{T} = T + K(X)$ and $\hat{T} = T + A(X)$.*

This is interesting in view of Enflo's result which assures the existence of Banach spaces X for which $A(X) \neq K(X)$.

For the sake of completeness we give explicitly some elementary definitions. Let X be a complete metric space and A a bounded subset of X. Following Kuratowski [5], we define $\gamma(A)$, which we shall call the measure of noncompactness of A, to be inf $\{d > 0; A$ can be covered by finitely many sets of diameter $\leq d\}$.

If A and X are as above and S is a nonempty subset of X, following Goldenstein, Gohberg and Markus [3] we define $\tilde{\gamma}_S(A)$, which we shall call the ball measure of non-compactness of A in S, to be inf $\{r > 0; A$ can be covered by finitely many balls with centers in S and radii $\leq$ r$\}$.

If $S = X$ we simply write $\tilde{\gamma}_X = \tilde{\gamma}$. The reason for this terminology is simple, since a complete metric space is compact iff it is totally bounded: $\gamma(A) = 0$ iff $\tilde{\gamma}(A) = 0$ iff A is relatively compact.

If X and Y are complete metric spaces and $f:X \to Y$ is a continuous function, we say that f is a k-set contraction if for every bounded set A in X, $\gamma_Y(f(A)) \leq k\gamma_X(A)$, and we say that f is a ball-k-set contraction if $\tilde{\gamma}_Y(f(A)) \leq k\tilde{\gamma}_X(A)$ for every bounded set A in X. We define

$$\gamma(f) = \inf \{k; f \text{ is a } k\text{-set contraction}\} \text{ and}$$

$$\tilde{\gamma}(f) = \inf \{k; f \text{ is a ball-}k\text{-set contraction}\}.$$

Let X be a normed space and Y be a Banach space. We denote by $L(X, Y)$ the Banach space of all bounded linear operators from X to Y, and by $K(X, Y)$ the Banach space of all compact linear operators from X to Y. If $X = Y$ we simply write $L(X) = L(X, X)$ and $K(X) = K(X, X)$. If X is a Banach space and if $\tilde{X} = L(X)/K(X)$ denotes the Calkin algebra together with the usual norm $\| \tilde{T} \| = \inf \{ \| T + K \| ; K \in K(X)\}$ $(T \in L(X))$, then it can be shown [7] that $\gamma(T) \leq \| \tilde{T} \|$.

Now, let T be a closed, densely defined linear operator on a Banach space X. F. E. Browder [1] defined the essential spectrum $\sigma_e(T)$ of T, to be the set of $\lambda \in \sigma(T)$, the spectrum of T, such that at least one of the following conditions hold: (1) $R(\lambda - T)$, the range of $\lambda - T$, is not closed; (2) λ is a limit point of $\sigma(T)$; (3) $\cup_{n\geq 1} N((\lambda - T)^n)$ is infinite dimensional, where $N(T)$ denotes the nullspace of a linear operator T. Browder proved that $\lambda_0 \notin \sigma_e(T)$ iff for some $\delta > 0$, λ is in the resolvent set of T for $0 < \| \lambda - \lambda_0 \| < \delta$ and the Laurent expansion of $(\lambda - T)^{-1}$ around λ_0 has only a finite number of non-zero coefficients with negative indices.

The main result in [7] is the following:

THEOREM. (Nussbaum): *Let X be a (complex) Banach space and $T \in L(X)$. If we define the essential spectral radius of T to be*

$$r_e(T) = \sup \{|\lambda|; \lambda \in \sigma_e(T)\},$$

then

$$r_e(T) = \lim_{n\to\infty} (\gamma(T^n))^{1/n} = \lim_{n\to\infty} (\tilde{\gamma}(T^n))^{1/n} = \lim_{n\to\infty} (\| \tilde{T}^n \|)^{1/n}$$

In particular, we see that the spectral radius of the element $\tilde{T}$ of the Banach algebra $\tilde{X}$ is precisely the essential spectral radius of the linear operator $T \in L(X)$.

2. Another characterization of $r_e(T)$

If X is a metric space we denote the open ball centered at $x \in X$ and radius $r > 0$ by $B_r(x)$. We let $B = B_1(0)$, if X is normed. The following concept was used by A. Pietsch in [8]: Let X and Y be normed spaces. For every integer $n \geq 0$ we let $F_n(X, Y) = \{T \in L(X, Y); \dim (T(X)) \leq n\}$, and $F(X, Y) = \cup_{n\geq 0} F_n(X, Y)$. Then $F(X, Y)$ is the vector subspace of $L(X, Y)$ consisting of all linear operators of finite rank. If $T \in L(X, Y)$ we define the approximation number of T of order n, as the number $\alpha_n(T) = \inf \{ \| T - F \| ; F \in F_n(X, Y)\}$. It is clear that $\| T \| = \alpha_0(T) \geq \alpha_1(T) \geq \cdots \geq \alpha_n(T) \geq \cdots$.

Now let $A(X, Y)$ be the closure of the set $F(X, Y)$ in $L(X, Y)$. Then we have

PROPOSITION 1: *Let X and Y be Banach spaces. For every $T \in L(X, Y)$ we define $\alpha(T) = \lim_{n\to\infty} \alpha_n(T)$.*
Then we have:
(a) $\alpha_n(T) = 0$ *iff* $T \in F_n(X, Y)$.
(b) $\alpha(T) = 0$ *iff T is approximable, in the operator norm, by linear operators of finite rank. In particular* $A(X, Y) = \{T \in L(X, Y); \alpha(T) = 0\} \subseteq K(X, Y)$.
(c) $\| \tilde{T} \| \leq \alpha_n(T)$ *for every* $n \geq 0$, *and hence* $\| \tilde{T} \| \leq \alpha(T) \leq \| T \|$.
(d) $T \in L(X, Y)$ *and* $F \in F_n(X, Y)$ *imply that* $\alpha_n(T + F) = \alpha_n(T)$, *and hence* $\alpha(T + F) = \alpha(T)$.

Proof: (a), (b) and (d) follow immediately from the definitions of α_n and α.
(c) We have from $F_n(X, Y) \subseteq K(X, Y)$ that $\| \tilde{T} \| = \inf \{ \| T + K \| ; K \in K(X, Y)\} \leq \inf \{ \| T - F \| ; F \in F_n(X, Y)\} = \alpha_n(T)$.

PROPOSITION 2: *Let X and Y be Banach spaces. Then:*
(a) *α is a continuous seminorm on $L(X, Y)$*
(b) $\alpha(T + S) = \alpha(T)$, *for every* $T \in L(X, Y)$ *and* $S \in A(X, Y)$.
(c) *If* $X = Y$ *and if* $S, T \in L(X)$, *then* $\alpha(ST) \leq \alpha(S)\, \alpha(T)$.

Proof: (a) In [8; 121–122] it is shown that $\alpha_n(\lambda T) = |\lambda|\alpha_n(T)$ and $\alpha_{m+n}(S + T) \leq \alpha_m(S) + \alpha_n(T)$. If we let $m, n \to \infty$ we obtain $\alpha(\lambda T) = |\lambda|\alpha(T)$ and $\alpha(S + T) \leq \alpha(S) + \alpha(T)$.
Thus α is a seminorm on $L(X, Y)$. That it is continuous follows from the obvious inequalities $|\alpha(S) - \alpha(T)| \leq \alpha(S - T) \leq \| S - T \|$.
(b) Let $T \in L(X, Y)$ and $S \in A(X, Y)$. Then there is a sequence $\{S_m\}$ in $F(X, Y)$ such that $\| S - S_m \| \to 0$ as $m \to \infty$, and since α is continuous we have from Proposition 1 (d) $\alpha(T + S) = \lim_{m\to\infty} \alpha(T + S_m) = \alpha(T)$.
(c) In [8; 122] it is shown that $\alpha_{m+n}(ST) \leq \alpha_m(S)\alpha_n(T)$. If we let $m, n \to \infty$ we obtain the desired result.

PROPOSITION 3: *Let X be a Banach space and let $T \in L(X)$.*
(a) *The limit* $\lim_{n\to\infty} (\alpha(T^n))^{1/n}$ *exists and equals* $\inf_{n>0} (\alpha(T^n))^{1/n}$ *If we denote this limit by* $a(T)$, *then* $a(T) \leq r_\sigma(T)$ *and* $a(T) \leq \alpha(T)$, *where* $r_\sigma(T)$ *denotes the spectral radius of T.*
(b) *If* $F \in F(X)$ *then* $a(T + F) = a(T)$.

Proof: (a) The proof that $\lim_{n\to\infty} (\alpha(T^n))^{1/n} = \inf_{n>0} (\alpha(T^n))^{1/n}$ is a standard argument (see, for example, [9; 212]). One only needs Proposition 2 (c) and the fact that $\alpha(T) \geq 0$ for $T \in L(X)$, to get the desired result. The inequalities are obvious.
(b) If $F \in F(X)$ then we can write $(T + F)^n = T^n + F_n$, where F_n is a linear operator of finite rank. Hence from Proposition 1 (d) we have $\alpha((T + F)^n) = \alpha(T^n + F_n) = \alpha(T^n)$, $n \geq 0$; and this implies that $a(T + F) = a(T)$.

PROPOSITION 4: *If X is a Banach space and* $T \in L(X)$, *then*

$$r_e(T) = \lim_{n\to\infty} (\alpha(T^n))^{1/n}$$

Proof: From Proposition 1 (c) we have $(\| \tilde{T}^n \|)^{1/n} \leq (\alpha(T^n))^{1/n}$ $(n > 0)$; and hence from Nussbaum's theorem and Proposition 3 (a) we obtain $r_e(T) \leq a(T)$. Given $r > r_e(T)$, it is shown in [7] (Lemma 6) that there is an $F \in F_n(X)$, for some $n \geq 0$, such that $TF = FT$ and $r_\sigma(T + F) \leq r$. Combining this with the inequality above and with Proposition 3 we obtain $r \geq r_\sigma(T + F) \geq a(T + F) = a(T) \geq r_e(T)$. Since we can take r arbitrarily close to $r_e(T)$, we conclude that $r_e(T) = a(T)$.

Now, recall that we have $A(X) = \{T \in L(X); \alpha(T) = 0\}$ (Proposition 1 (b)), $A(X) \subseteq K(X)$ and $A(X)$ is a closed two sided ideal of the Banach algebra $L(X)$. Consider the quotient algebra $\hat{X} = L(X)/A(X)$, together with the norm $\| \hat{T} \| = \inf \{ \| T + S \| ; S \in A(X)\}$, where $\hat{T} = T + A(X)$. Then $\hat{X}$ is a Banach algebra, and if $A(X) = K(X)$, then $\tilde{X} = \hat{X}$.

PROPOSITION 5: *Let $T \in L(X)$. Then*:
(a) $\| \hat{T} \| = \alpha(T)$.
(*b*) $lim_{n\to\infty} (\| \hat{T}^n \|)^{1/n} = \lim_{n\to\infty} (\| \tilde{T}^n \|)^{1/n} = r_e(T)$. i.e., *the spectral radius of $\tilde{T} \in \tilde{X}$ and $\hat{T} \in \hat{X}$ coincide.*

Proof: (a) Since $\alpha(T) = \inf \{ \| T - F \| ; F \epsilon F(X)\}$ and $F(X) \subseteq A(X)$, then we have $\alpha(T) \geq \| \hat{T} \|$. Now, let $\eta > \| \hat{T} \|$ and $\epsilon > 0$ such that $\eta > \epsilon > \| \hat{T} \|$.

Then there is a $S \in A(X)$ such that $\| T - S \| < \frac{\epsilon}{2}$. But $S \epsilon A(X)$ implies that

there is a $U \in F(X)$ such that $\| S - U \| < \frac{\epsilon}{2}$.
Adding these two inequalities we obtain $\alpha(T) \leq \| T - U \| \leq \| T - S \| + \| S - U \| < \epsilon < \eta$. Since we can choose η arbitrarily close to $\| \hat{T} \|$ we conclude that $\| \hat{T} \| \geq \alpha$(T).
(b) immediate from (a) and Proposition 4.

3. **Some related results**

We start with the following concept due to Kolmogoroff [4]: Let X be a normed space and F a subspace of X. If A is a bounded subset of X and $n \geq 0$ is an integer we define $\delta_n(F; A)$, which we shall call the n-diameter of A in F, to be inf $\{\delta > 0; A \subseteq \delta B + G$, where G is a subspace of F with dimension $\leq n\}$. It is clear that $\delta_0(F; A) \geq \delta_1(F; A) \geq \cdots \geq \delta_n(F; A) \geq \cdots$. If $F = X$ we simply write $\delta_n(A) = \delta_n(X; A)$, $n \geq 0$.

PROPOSITION 6: *Let X be a Banach space and F a substance of X. If for every bounded set A in X we let $\delta(F; A) = \lim_{n\to\infty} \delta_n(F; A)$, then $\delta(F; A) = \tilde{\gamma}_F(A)$.*

Proof: Let $r > \tilde{\gamma}_F(A)$, then there are points $x_1, \cdots, x_n \epsilon F$ such that $A \subseteq \cup_{i=1}^n (x_i + rB)$. If we let G be the subspace spanned by the vectors $x_1 \cdots, x_n$, then $G \subseteq F$, dim $(G) \leq n$ and $A \subseteq rB + G$. Thus $\delta(F; A) \leq \delta_n(F; A) \leq r$; and since r is arbitrary we must have $\delta(F; A) \leq \tilde{\gamma}_F(A)$.

Now let $\epsilon > 0$ be given and let $\alpha = \delta(F; A) + \epsilon$. Then there is an $n \geq 0$ such that $\delta_n(F; A) < \alpha$. Hence there exists a subspace G of F with dim $(G) \leq n$ such that $A \subseteq \alpha B + G$. Let $x \epsilon A$, then we can write $x = \alpha y + z$, $y \epsilon B$, $z \epsilon G$. Thus $\| z \| \leq \alpha \| y \| + \| x \| \leq \alpha + \| x \|$; and since $x \epsilon A \subseteq \beta B$ for some $\beta > 0$ large enough, we must have $\| z \| \leq \alpha + \beta$ i.e., $z \epsilon U = G \cap (\alpha + \beta)B$. Since U is totally bounded in G there exists points $x_1, \cdots, x_n \epsilon U$ such that $U \subseteq \cup_{i=1}^{n} (x_i + \epsilon(B \cap G)) \subseteq \cup_{i=1}^{n} (x_i + \epsilon B)$. Thus $x = \alpha y + z \epsilon \alpha B + U \subseteq \cup_{i=1}^{n} (x_i + (\alpha + \epsilon)B)$, where $x_1, \cdots, x_n \epsilon F$. Hence $\tilde{\gamma}_F(A) \leq \alpha + \epsilon = \delta(F; A) + 2\epsilon$. Since $\epsilon > 0$ is arbitrary our proof is complete.

The following result relates the concepts of approximation numbers of order n, n-diameters and ball measure of noncompactness.

PROPOSITION 7: *Let X and Y be Banach spaces and let $T \epsilon L(X, Y)$. Denote by B_x and B_y the unit balls in X and Y respectively.*
Then:
(a) $\alpha_n(T) \geq \delta_n(T(B_x))(n \geq 0)$, *and hence* $\alpha(T) \geq \delta(T(B_x)) = \tilde{\gamma}(T(B_x)) = \tilde{\gamma}(T)$.
(b) *If Y is a Hilbert space, then* $\alpha_n(T) = \delta_n(T(B_x))$ $(n \geq 0)$, *and hence* $\alpha(T) = \delta(T(B_x)) = \tilde{\gamma}(T(B_x)) = \tilde{\gamma}(T)$.

Proof: (a) from [7; 474] we have $\tilde{\gamma}(T) = \tilde{\gamma}(T(B_x))$. Thus from Proposition 6 we have $\delta(T(B_x)) = \tilde{\gamma}(T(B_x)) = \tilde{\gamma}(T)$. From [8; 148] we obtain $\delta_n(T(B_x)) \leq \alpha_n(T)$, and hence $\delta(T(B_x)) \leq \alpha(T)$.
(*b*) *Let* $\rho > \delta_n(T(B_x))$, then there is a $\delta > 0$ with $\delta_n(T(B_x)) \leq \delta < \rho$ and a subspace G of H with dim $(G) \leq n$ such that $T(B_x) \subseteq \delta B_y + G$. Let $P: Y \to G$ the orthogonal projection. Given $x \epsilon B_x$ we can write $T(x) = \delta y + z$, $y \epsilon B_y$, $z \epsilon G$. Since $T(x) - PT(x)$ is orthogonal to $PT(x) - z$ we obtain $\delta^2 \geq \| \delta y \|^2 = \| T(x) - z \|^2 = \| PT(x) - z \|^2 + \| T(x) - PT(x) \|^2 \geq \| T(x) - PT(x) \|^2$, $x \epsilon B_x$. Thus we have $\| T - PT \| \leq \delta$, where $PT \epsilon F_n(X, Y)$; and hence $\alpha_n(T) \leq \| T - PT \| \leq \delta < \rho$. Since ρ is arbitrary we conclude that $\alpha_n(T) \leq \delta_n(T(B_x))$.

Since $T \epsilon L(X, Y)$ is compact iff $\tilde{\gamma}(T) = 0$, then we see that the identity $\alpha(T) = \tilde{\gamma}(T)$ in Proposition 6 (b) is a generalization of the following well known result: Let X be a Banach space and Y be Hilbert space. If $T \epsilon L(X, Y)$, then T is compact iff it can be approximated, in the operator norm, by linear operators of finite rank.

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