LIMIT THEOREM FOR CERTAIN RANDOM MOTIONS OF R^d

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1. Introduction

We consider in this paper a general type of random motions of Euclidean space R^d that are motivated by certain physical phenomena. The object of the paper is to obtain a functional central limit theorem for such random motions. In Section 2 we define the motions, and state the limit theorem; this result contains as a special case the central limit theorem for random displacements of R^d ([18], [16], [9]). Section 3 concerns the proof of the theorem. In Section 4 we consider the physical problem that led us to this investigation, and discuss how the limit theorem may possibly yield approximations of boundary crossing probabilities for certain problems in seismological engineering.

2. Random motions and limit theorem

Let

R^d = d-dimensional Euclidean space ($d \geq 1$),

$\mathscr{G}$ = the group of rigid motions of R^d,

$\mathscr{U}$ = the group of (proper and improper) rotations of R^d,

$\mathscr{V}$ = the group of parallel translations of R^d (isomorphic to R^d).

It is well-known that an element $g \in \mathscr{G}$ may be written as $g = v(g)u(g)$ (with $v(g)$ acting first), where $u(g) \in \mathscr{U}$ and $v(g) \in \mathscr{V}$. If we represent translations $v \in \mathscr{V}$ by column-vectors V, and rotations $u \in \mathscr{U}$ by orthogonal matrices U, then the product $g(n) = g_1 \cdots g_n$, with $g_i = v_i u_i$, $u_i \in \mathscr{U}$, $v_i \in \mathscr{V}$, $i = 1, \cdots, n$, is written as $g(n) = v(n)u(n)$, with $u(n) \in \mathscr{U}$ and $v(n) \in \mathscr{V}$ represented respectively by the matrix $U(n) = U_1 \cdots U_n$, and the vector $V(n) = \Sigma_{i=1}^{n} U_0 \cdots U_{i-1}V_i$ (U_0 is the identity matrix I, henceforth omitted in such expressions).

Let $g_i \equiv \{g_i(t),\ t \geq 0\}$, $i = 1, 2, \cdots$, be a sequence of $\mathscr{G}$-valued random functions that are independent and identically distributed (i.i.d.). (R^d has the norm $\| x \| = (\Sigma_{i=1}^{d} x_i^2)^{1/2}$, $x = (x_1, \cdots, x_d)$; $\mathscr{G}$ is topologized by the operator norm; $\mathscr{G}$ is separable and complete). We assume that the trajectories of the rotation and translation components of the g_i are right-continuous and have left limits everywhere. Let T_i, $i = 1, 2, \cdots$, be random times, i.i.d., and independent of the g_i. A probability space $(\Omega, \mathscr{F}, P)$ exists on which all this is defined.

On $(\Omega, \mathscr{F}, P)$ we define a $\mathscr{G}$-valued random function $\phi \equiv \{\phi(t),\ t \geq 0\}$ as follows. Let $S_0 = 0$, $S_n = \Sigma_{i=1}^{n} T_i$, $n \geq 1$, and $N(t) = \max\{n : S_n \leq t\}$, $t \geq 0$; then

$$\phi(t) = g_1(T_1)g_2(T_2) \cdots g_{N(t)}(T_{N(t)})g_{N(t)+1}(t - S_{N(t)}), \quad t \geq 0$$

(which is interpreted as $\phi(t) = g_1(t)$ for $t < T_1$, a convention maintained throughout). ϕ means that g_1 acts on R^d from time 0 to time S_1, and for $j > 1$, g_j acts on $R^d g_1(T_1) \cdots g_{j-1}(T_{j-1})$ from time S_{j-1} to time S_j.

Using the above representations, we write

$$\phi(t) = v(t)u(t),$$

with $u(t) \in \mathscr{U}$ and $v(t) \in \mathscr{V}$ represented respectively by the matrix

$$U(t) = U_1(T_1) \cdots U_{N(t)}(T_{N(t)})U_{N(t)+1}(t - S_{N(t)}),$$

and the vector

$$V(t) = \Sigma_{i=1}^{N(t)} U_1(T_1) \cdots U_{i-1}(T_{i-1})V_i(T_i) + U_1(T_1) \cdots U_{N(t)}(T_{N(t)})V_{N(t)+1}(t - S_{N(t)}),$$

where $U_i(t)$ and $V_i(t)$ are the representations of $u_i(t)$ and $v_i(t)$ such that $g_i(t) = v_i(t)u_i(t)$. It is easy to see that the trajectories of the processes $U \equiv \{U(t), t \geq 0\}$ and $V \equiv \{V(t), t \geq 0\}$ are right-continuous with left limits.

Our aim is to establish a functional central limit theorem for the translation process V, for small durations of the g_i actions. For this purpose, we consider the sequence

$$\{\phi^n(t) = v^n(t)u^n(t), t \geq 0\}, \quad n = 1, 2, \cdots,$$

corresponding to the following normalizations: T_i is replaced by T_i/n (hence $N(t)$ becomes $N(nt)$), and the translation component is multiplied by $n^{1/2}$. Thus, $u^n(t) \in \mathscr{U}$ and $v^n(t) \in \mathscr{V}$ are represented respectively by the matrix

$$U^n(t) = U^*_{n,1} \cdots U^*_{n,N(nt)}U^*_n(t), \quad t \geq 0,$$

and the vector

$$V^n(t) = n^{1/2} \Sigma_{i=1}^{N(nt)} U^*_{n,1} \cdots U^*_{n,i-1}V^*_{n,i} + n^{1/2}U^*_{n,1} \cdots U^*_{n,N(nt)}V^*_n(t), \quad t \geq 0,$$

where

$$U^*_{n,i} = U_i(T_i/n), \quad V^*_{n,i} = V_i(T_i/n), \quad i \geq 1;$$

$$U^*_n(t) = U_{N(nt)+1}(t - S_{N(nt)}/n), \quad V^*_n(t) = V_{N(nt)+1}(t - S_{N(nt)}/n), \quad t \geq 0.$$

We assume the following conditions:

1. $P[U_1(0)W \subseteq W] < 1$ for all nontrivial subspaces W of R^d, if $d \geq 2$, and $P[U_1(0) = 1] < 1$ if $d = 1$.

2. $E \sup_{0 \leq s \leq t} \| U_1(s) - U_1(0)\|^q \leq (Kt^{(\delta+3/2)})^q$, for each t and $q \geq 1$, and some δ, $0 < \delta \leq \frac{1}{2}$.

3. $V_i(t)$ is twice continuously differentiable with bounded second derivative uniformly over Ω, $\dot{V}_i(0) = 0$ (hence there are no translation jumps), and $E \| \dot{V}_1(0)\|^{2+\delta} < \infty$ for some $\delta > 0$ ($\dot{}$ denotes right-derivative with respect to time).

4. T_i is not identically 0 (but may have an atom at 0), and $ET_1^{3+\delta} < \infty$ for some $\delta > 0$.

5. $\{E \sup_{0 \leq s \leq t} \| V_1(s)/s \|^2\}_t$ is uniformly bounded.

Without an irreducibility requirement such as Condition 1, the process ϕ may concentrate on a proper subspace of R^d, if $d \geq 2$, or not change direction, if $d = 1$. Condition 1 implies that $I - EU_1(0)$ is nonsingular ([9], Lemma 1). Some of the requirements in the other conditions are stronger than necessary, in order to simplify calculations here.

To simplify the notation in the theorem, we denote $U = U_1(0)$, $\dot{V} = \dot{V}_1(0)$, and $T = T_1$.

Some final notation. $\Rightarrow$ means weak convergence, B is standard d-dimensional Brownian motion, tr, $^{-1}$, and ^{t} are trace, inverse, and transpose of a matrix.

Under the above conditions, we have the following functional central limit theorem.

THEOREM. *$V^n \Rightarrow \sigma B$ as $n \to \infty$, where*

$$\sigma^2 = \frac{1}{dET} \{E \| \dot{V} \|^2 \, Var\, T + (2tr[(I - EU)^{-1}(E\dot{V})(E\dot{V}^t U)] + E \| \dot{V} \|^2)(ET)^2\}.$$

Remarks.

1) $\Rightarrow$ refers to weak convergence of probability measures on $D[0, \infty)^d$ with the Skorohod topology (see [11], and references in [8], [9]). This space is needed in the proof.

2) With $T \equiv 1$, the theorem contains the functional central limit theorem of [9], and σ^2 has the same form.

3) The special case $T \equiv 1$ and $t = 1$ yields a central limit theorem that contains the results obtained by Tutubalin [18] in 1967 for $d = 2$ and 3, and by Roynette [16] in 1974, for $d \geq 3$, both under the condition that the closed subgroup generated by the support of the rotation distribution is $SO(d)$, which implies that rotations converge, and without identifying σ^2. These results are also special cases of the theorem obtained by the author [9] (1973), as a special case of a functional central limit theorem, which itself is a special case of the present theorem. Moreover, to prove the convergence of the translation components it is not necessary to use the convergence of the rotation components, and in fact the rotations may not converge (see [9]).

4) Observe that for $d = 2$ the present process and that of [8] have a common special case, but in [8] σ^2 has a different expression; it is easy to verify that the two are equal.

5) Other works of related interest are [13] (random products of random matrices), and [1], [3], [4], [5], [6], [17], which contain other recent results on random displacements of R^d.

3. Proof.

The proof employs basically the same methods of [7], [8], [9], [10]. The main thing to notice is that V^n has essentially the same space structure as the process in [9], and the same time structure as the process in [8]; but there are also differences (e.g., in [8] position was a piecewise linear function of time, and now it may be nonlinear). Therefore our procedure consists in first preparing the problem so that the previous methods may be used, and then in combining those methods in an appropriate way. The preparation consists in replacing $\{V^n\}$ by simpler processes $\{\tilde{V}^n\}$ that converge to the same limit, and to which the methods apply. Once this is done, a consideration of the proofs of both [8] and [9] convinces one that $\{\tilde{V}^n\}$, and hence also $\{V^n\}$, converges weakly to σB, with some constant σ; then it remains to compute σ^2, which is actually the only thing that we need to do. Here we will carry out the replacement of $\{V^n\}$ by $\{\tilde{V}^n\}$, and leave the reader to see how to combine the techniques of the previous works to obtain σ^2. For the sake of brevity, we also let the reader think about the function spaces, topologies, and weak convergence theory that are relevant to the proof (see [7], [8], [9], and bibliography therein, specially [2]); therefore we make no explicit references to the things we are using.

Let

$$\tilde{U}^n(t) = U_1(0) \cdots U_{N(nt)+1}(0), \qquad t \geq 0,$$

$$\tilde{V}^n(t) = n^{-1/2} \textstyle\sum_{i=1}^{N(nt)} U_1(0) \cdots U_{i-1}(0) \dot{V}_i(0) T_i, \qquad t \geq 0.$$

LEMMA. $(U^n, V^n) \Rightarrow$ *if and only if* $(\tilde{U}^n, \tilde{V}^n) \Rightarrow$, *and in case of convergence the limits coincide.*

Proof. It suffices to show that for any $\epsilon > 0$ and $\tau > 0$,
a) $P[\sup_{0\leq t\leq\tau} \| U^n(t) - \tilde{U}^n(t)\| \geq \epsilon] \to 0$ as $n \to \infty$, and
b) $P[\sup_{0\leq t\leq\tau} \| V^n(t) - \tilde{V}^n(t)\| \geq \epsilon] \to 0$ as $n \to \infty$.

Proof of a): We have

$$\| U^n(t) - \tilde{U}^n(t)\|$$
$$= \| U^*_{n,1} \cdots U^*_{n,N(nt)} U^*_n(t) - U_1(0) \cdots U_{N(nt)}(0) U_{N(nt)+1}(0)\|$$
$$\leq \textstyle\sum_{i=1}^{N(nt)} \| U_i(T_i/n) - U_i(0)\| + \| U_{N(nt)+1}(t - S_{N(nt)}/n) - U_{N(nt)+1}(0)\|,$$

where we used the fact that rotations have norm 1. Hence

$$P[\sup_{0\leq t\leq\tau} \| U^n(t) - \tilde{U}^n(t)\| \geq 2\epsilon]$$
$$\leq P[\sup_{0\leq t\leq\tau} \| U_{N(nt)+1}(t - S_{N(nt)}/n) - U_{N(nt)+1}(0)\| \geq \epsilon]$$
$$+ P[\textstyle\sum_{i=1}^{N(n\tau)} \| U_i(T_i/n) - U_i(0)\| \geq \epsilon].$$

For the first term on the right, we have

$$P[\sup_{0\leq t\leq\tau} \| U_{N(nt)+1}(t - S_{N(nt)}/n) - U_{N(nt)+1}(0)\| \geq \epsilon]$$
$$\leq P[\max_{1\leq k\leq N(n\tau)+1} \sup_{S_{k-1}/n\leq t\leq S_k/n} \| U_k(t - S_{k-1}/n) - U_k(0)\| \geq \epsilon];$$

since $N(t)/t \to 1/ET_1$ a.s. as $t \to \infty$, then (as in [7], Proposition 1), denoting $\nu = 1/ET_1$, for $\delta > 0$ we have

$$\limsup_n P[\max_{1\le k\le N(n\tau)+1} \sup_{S_{k-1}/n\le t\le S_k/n} \| U_k(t - S_{k-1}/n) - U_k(0)\| \ge \epsilon]$$

$$\le 1 - \liminf_n (1 - P[\sup_{0\le t\le T_1/n} \| U_1(t) - U_1(0)\| \ge \epsilon])^{n(\nu\tau+\delta)}$$

$$\cdot P[\sup_{0\le t\le T_1/n} \| U_1(t) - U_1(0)\| < \epsilon];$$

but

$$P[\sup_{0\le t\le T_1/n} \| U_1(t) - U_1(0)\| < \epsilon] \to 1,$$

by right-continuity of rotations; and

$$(1 - P[\sup_{0\le t\le T_1/n} \| U_1(t) - U_1(0)\| \ge \epsilon])^{n(\nu\tau+\delta)} \to 1,$$

because

$$P[\sup_{0\le t\le T_1/n} \| U_1(t) - U_1(0)\| \ge \epsilon] = 0(n^{-1});$$

indeed,

$$nP[\sup_{0\le t\le T_1/n} \| U_1(t) - U_1(0)\| \ge \epsilon]$$

$$= n \int_0^\infty P[\sup_{0\le t\le x/n} \| U_1(t) - U_1(0)\| \ge \epsilon] P[T_1 \in dx]$$

$$\le n\epsilon^{-1} \int_0^\infty E \sup_{0\le t\le x/n} \| U_1(t) - U_1(0)\| P[T_1 \in dx]$$

$$\le K\epsilon^{-1} n^{-(\delta+1/2)} E T_1^{\delta+3/2} \quad \text{(Condition 2);}$$

the last part goes to 0 as $n \to \infty$.

For the second term,

$$P[\Sigma_{i=1}^{N(n\tau)} \| U_i(T_i/n) - U_i(0)\| \ge \epsilon] \le \epsilon^{-1} E\, \Sigma_{i=1}^{N(n\tau)} \| U_i(T_i/n) - U_i(0)\|$$

$$= \epsilon^{-1} EN(n\tau) \| U_1(T_1/n) - U_1(0)\| \qquad \text{(use [7], Corollary 2)}$$

$$\le \epsilon^{-1}[EN(n\tau)^p]^{1/p}[E \| U_1(T_1/n) - U_1(0)\|^q]^{1/q} \qquad \text{(Hölder, } 1/p + 1/q = 1\text{)}$$

$\le \epsilon^{-1}[EN(n\tau)^p]^{1/p} K n^{-(\delta+3/2)}[E T_1^{(\delta+3/2)q}]^{1/q}$ (Condition 2, p large integer, and δ such that $(\delta + 3/2)q \le 3$)

$$= \epsilon^{-1} K[E(N(n\tau)/n)^p]^{1/p} n^{-(\delta+1/2)}[E T_1^{(\delta+3/2)q}]^{1/q};$$

the last expression goes to 0 as $n \to \infty$, because $\{E(N(n\tau)/n)^p\}_n$ is bounded (see [7]).

Proof of b): We have

$$\| V^n(t) - \tilde{V}^n(t)\|$$

$$\le n^{1/2} \| V_n^*(t)\|$$

$$+ n^{1/2} \| \Sigma_{i=1}^{N(nt)} (U_{n,1}^* \cdots U_{n,i-1}^* V_{n,i}^* - U_1(0) \cdots U_{i-1}(0) \dot{V}_i(0) T_i/n)\|$$

$$\leq n^{1/2}\| V_{N(nt)+1}(t - S_{N(nt)}/n)\|$$
$$+ n^{1/2} \textstyle\sum_{i=1}^{N(nt)} \| V_i(T_i/n) - \dot{V}_i(0)T_i/n \|$$
$$+ n^{-1/2} \textstyle\sum_{i=1}^{N(nt)} \sum_{j=1}^{i-1} \| U_j(T_j/n) - U_j(0)\| \| \dot{V}_i(0)\| T_i.$$

Hence

$$P[\sup_{0\leq t\leq\tau} \| V^n(t) - \tilde{V}^n(t)\| \geq 3\epsilon]$$
$$\leq P[\sup_{0\leq t\leq\tau} n^{1/2}\| V_{N(nt)+1}(t - S_{N(nt)}/n)\| \geq \epsilon]$$
$$+ P[n^{1/2} \textstyle\sum_{i=1}^{N(n\tau)} \| V_i(T_i/n) - \dot{V}_i(0)T_i/n \| \geq \epsilon]$$
$$+ P[n^{-1/2} \textstyle\sum_{i=1}^{N(n\tau)} \sum_{j=1}^{i-1} \| U_j(T_j/n) - U_j(0)\| \| \dot{V}_i(0)\| T_i \geq \epsilon].$$

For the first term, similarly as above,

$$\limsup_n P[\sup_{0\leq t\leq\tau} n^{1/2}\| V_{N(nt)+1}(t - S_{N(nt)}/n)\| \geq \epsilon]$$
$$\leq 1 - \liminf_n (1 - P[\sup_{0\leq t<T_i/n} n^{1/2}\| V_1(t)\| \geq \epsilon])^{n(\nu\tau+\delta)}$$
$$\cdot P[\sup_{0\leq t\leq T_1/n} n^{1/2}\| V_1(t)\| < \epsilon]$$
$$\leq 1 - \liminf_n (1 - P[\sup_{0\leq t\leq T_1/n} \| V_1(t)/t\| T_1 \geq n^{1/2}\epsilon])^{n(\nu\tau+\delta)}$$
$$\cdot P[\sup_{0\leq t\leq T_1/n} \| V_1(t)/t\| T_1 < n^{1/2}\epsilon],$$

which is 0 because $P[\sup_{0\leq t\leq T_1/n} \| V_1(t)/t\| T_1 < n^{1/2}\epsilon] \to 1$ (Condition 3), and

$$P[\sup_{0\leq t\leq T_1/n} \| V_1(t)/t\| T_1 \geq n^{1/2}\epsilon] = o(n^{-1}),$$

since

$$nP[\sup_{0\leq t\leq T_1/n} \| V_1(t)/t\| T_1 \geq n^{1/2}\epsilon]$$
$$= n \textstyle\int_0^\infty P[\sup_{0\leq t\leq x/n} \| V_1(t)/t\| x \geq n^{1/2}\epsilon]P[T_1 \in dx]$$
$$\leq \epsilon^{-2} \textstyle\int_0^\infty (\int_{A_n} \sup_{0\leq t\leq x/n} \| V_1(t)/t\|^2 \, dP)x^2P[T_1 \in dx],$$

where $A_n = [\sup_{0\leq t\leq x/n} \| V_1(t)/t\| x \geq n^{1/2}\epsilon]$, and the last expression goes to 0 as $n \to \infty$, due to Conditions 3, 4 and 5, and the dominated convergence theorem.

For the second term,

$$P[n^{1/2} \textstyle\sum_{i=1}^{N(n\tau)} \| V_i(T_i/n) - \dot{V}_i(0)T_i/n \| \geq \epsilon] \leq P[\sum_{i=1}^{N(n\tau)} T_i^2 \geq M\epsilon n^{3/2}]$$

(Taylor, and Condition 3, M is a constant)

$$\leq (M\epsilon n^{3/2})^{-1}E \textstyle\sum_{i=1}^{N(n\tau)} T_i^2 = (M\epsilon n^{3/2})^{-1}EN(n\tau)T_1^2 \quad \text{(use [7], Corollary 2)}$$

$$= (M\epsilon)^{-1}n^{-1/2}ET_1^2N(n\tau)/n,$$

which goes to 0 as $n \to \infty$, because $\{ET_1^2N(n\tau)/n\}_n$ is bounded.

For the third term,

$$P[n^{-1/2} \Sigma_{i=1}^{N(n\tau)} \Sigma_{j=1}^{i-1} \| U_j(T_j/n) - U_j(0)\| \| \dot{V}_i(0)\| T_i \geq \epsilon]$$
$$\leq (\epsilon n^{1/2})^{-1} E \, \Sigma_{i=1}^{N(n\tau)} \Sigma_{j=1}^{i-1} \| U_j(T_j/n) - U_j(0)\| \| \dot{V}_i(0)\| T_i$$
$$\leq (\epsilon n^{1/2})^{-1} EN(n\tau)^2 \| U_1(T_1/n) - U_1(0)\| T_2 E \| \dot{V}_1(0)\|$$

(independence, and [7], Corollary 2)

$$\leq (\epsilon n^{1/2})^{-1} [EN(n\tau)^{2p}]^{1/p} [E \| U_1(T_1/n) - U_1(0)\|^q]^{1/q} (ET_1{}^q)^{1/q} E \| \dot{V}_1(0)\|$$

(Hölder, $1/p + 1/q = 1$, and independence)

$$\leq K\epsilon^{-1} [EN(n\tau)^{2p}]^{1/p} n^{-(\delta+2)} (ET_1{}^{(\delta+3/2)q})^{1/q} (ET_1{}^q)^{1/q} E \| \dot{V}_1(0)\|$$

(Condition 2, p large integer, and δ such that $(\delta + 3/2)q \leq 3$)

$$= K\epsilon^{-1} [E(N(n\tau)/n)^{2p}]^{1/p} n^{-\delta} (ET_1{}^{(\delta+3/2)q})^{1/q} (ET_1{}^q)^{1/q} E \| \dot{V}_1(0)\|,$$

which goes to 0 as $n \to \infty$.

The lemma is proved.

Observe that $V^n \Rightarrow$ if and only if $\tilde{V}^n \Rightarrow$, and in case of convergence the limits coincide. Hence we need only to study the process

$$\tilde{V}^n(t) = n^{-1/2} \Sigma_{i=1}^{N(nt)} U_1 \cdots U_{i-1} \dot{V}_i T_i, \qquad t \geq 0,$$

where we have denoted $U_j = U_j(0)$, and $\dot{V}_j = \dot{V}_j(0)$. This is done by combining appropriately the methods of [8] and [9], and after a certain amount of work, the conclusion is that $\tilde{V}^n \Rightarrow \sigma B$, where σ^2 is as given in the theorem. (Observe the similarities of the roles of $\dot{V}_j$ here and T_j in [9], and T_j here and τ_j in [8]).

Notice that the convergence of $\{V^n\}$ is proved without recourse to convergence of $\{U^n\}$.

4. An application to seismology

Here we discuss a simple application of the limit theorem to a problem in seismological engineering. We only wish to put forth the general idea. The practical application of this idea for engineering use would require further investigation on the rates of convergence.

The following problem occurs in engineering. The motion of the ground due to an earthquake is modelled by a stochastic acceleration $Z \equiv \{Z(t), t \geq 0\}$, the ground motion induces a random motion $X \equiv \{X(t), t \geq 0\}$ with respect to the ground at a given point in a structure, and this phenomenon is represented by an equation of the form $AX(t) = -Z(t)$, $t \geq 0$, where A is a differential operator. It is desired to obtain information about the boundary crossing problem for X, i.e., about the probability that some appropriate function of the displacement and velocity of the given point in the structure shall exceed, in a given time interval, a level or boundary function beyond which damage might occur. One approach is to do experimental analysis, using for Z data of past earthquakes, which it is hoped are representative of future earthquakes in a given area.

Another approach is to do mathematical analysis, using for Z a simple random process (or generalized random process, such as white noise) that permits to solve the differential equation; in this case Z should contain relevant statistical information about earthquakes in a given area. It would be more satisfactory if Z resembled earthquake ground acceleration as much as possible, and the differential equation could be solved. This is not possible in general, but approximations can be obtained by means of the limit theorem, in the following way. We describe the earthquake acceleration by a sequence of processes $\{Z_n\}$ such that (with a normalization included) Z_n converges to a process Z in some sense; here Z_n represents the "real" earthquake ground acceleration, and Z is an asymptotic idealization. Suppose also that the corresponding sequence of responses $\{X_n\}$, which satisfy $AX_n = -Z_n$, converges weakly: $X_n \Rightarrow X$, and that the differential equation holds in the limit: $AX = -Z$. Suppose further that this limit differential equation can be solved. Then, by the theory of weak convergence, not only X is an approximation of the "real" response X_n, but also the solution of the boundary crossing problem for X is an approximate solution of the boundary crossing problem for X_n.

Concerning earthquake models, it seems, from literature on seismology (e.g. [12], [14], [15]), that a certain type of earthquake ground motion may be represented adequately as a succession of i.i.d. random velocity waves of i.i.d. random durations. This leads us to consider the following random motion of R^d. Let F_0 be a fixed frame of reference; we will follow the displacement of the origin with respect to F_0. Let $\{U_1(t),\ V_1(t),\ 0 \le t \le T_1\}$ be the first wave; $V_1(t)$ is translation of the origin, and $U_1(t)$ is (proper) rotation about the moving origin. At time T_1 the frame is rotated by $U_1(T_1)$ and its origin is at $V_1(T_1)$; let F_{T_1} denote this displaced frame. The second wave, $\{U_2(t),\ V_2(t),\ 0 \le t \le T_2\}$, acts on F_{T_1} in the same way as the first wave on F_0; hence, referred to F_0, the displacement of the origin under the second wave is given by $V_1(T_1) + U_1(T_1)V_2(t - T_1)$, $T_1 \le t \le T_1 + T_2$. And so on. Therefore the motion of the origin is given by the translation component V of ϕ studied above. Recall that V^n denotes the normalized process $n^{1/2}V$ with the T_i divided by n. So, if our structure is placed at the origin, we must study $AX_n = -\dot{V}^n$ (right derivative). By the limit theorem we know that $V^n \Rightarrow \sigma B$, and we expect that $X_n \Rightarrow X$, and that X satisfies $AX = -\sigma\dot{B}$, where $\dot{B}$ denotes white noise, and σ^2 is given in the theorem.

For a specific illustration we assume the earthquake takes place in the plane R^2, and we consider a one-dimensional linear oscillator excited by $\dot{V}_1{}^n$ (one component of $\dot{V}^n$). The oscillator's displacement relative to the ground, X_n, satisfies

$$\ddot{X}_n + 2h\dot{X}_n + p_0{}^2X_n = -\dot{V}_1{}^n, \qquad X_n(0) = \dot{X}_n(0) = 0,$$

where $h \ge 0$ and $p_0 > 0$ are constants. The solution of the differential equation is

$$X_n(t) = p^{-1} \textstyle\int_0^t e^{-h(t-s)}[h \sin p(t-s) - p \cos p(t-s)]V_1{}^n(s)\, ds,$$

$$\dot{X}_n(t) = \textstyle\int_0^t e^{-h(t-s)}[(p - h^2p^{-1}) \sin p(t-s) + 2h \cos p(t-s)]V_1{}^n(s)\, ds - V_1{}^n(t),$$

where $p = p_0(1 - h^2p_0{}^{-2})^{1/2}$. (In the undamped case, $h = 0$ and $p = p_0$).

We know that $V_1^n \Rightarrow \sigma B$, where B is now one-dimensional standard Brownian motion. X_n and $\dot{X}_n$ are continuous functions of V_1^n, therefore $X_n \Rightarrow X$ and $\dot{X}_n \Rightarrow \dot{X}$, where (by integration by parts)

$$X(t) = -\sigma p^{-1} \textstyle\int_0^t e^{-h(t-s)} \sin p(t-s)\, dB(s),$$

$$\dot{X}(t) = \sigma \textstyle\int_0^t e^{-h(t-s)}[hp^{-1} \sin p(t-s) - \cos p(t-s)]\, dB(s).$$

Hence X satisfies

$$\ddot{X} + 2h\dot{X} + p_0^2 X = -\sigma\dot{B}, \qquad X(0) = \dot{X}(0) = 0.$$

This equation is, except for the parameter σ, essentially the same one as that considered in [15], but here, rather than being proposed as a mathematical model for the oscillator excited by an earthquake ground acceleration idealized as white noise, the equation represents an asymptotic model that approximates the "real" situation. Moreover, the acceleration $\sigma\dot{B}$ contains in σ the precise and necessary information about the earthquakes that is needed in this model; this information should be obtained from earthquake ground motion statistics (or methods for obtaining it may be devised).

Observe that the limiting earthquake velocity process, σB, depends on the velocity waves only through expectations of random variables that are determined by information from the "starts" of the waves ($\dot{V}$ and U in the expression of σ^2), and the exact shapes of the whole waves are irrelevant.

It is of interest in engineering to study the boundary crossing problem for the random function (see [15]) $R = [(pX)^2 + (\dot{X} + hX)^2]^{1/2}$. Therefore we consider the process $R_n = [(pX_n)^2 + (\dot{X}_n + hX_n)^2]^{1/2}$. Clearly, $R_n \Rightarrow R$, and in particular, for fixed $\tau > 0$ and a (continuous) boundary function f,

$$P[\sup_{0 \le t \le \tau}(R_n(t) - f(t)) > 0] \to P[\sup_{0 \le t \le \tau}(R(t) - f(t)) > 0]$$

(if the underlying Borel set has boundary of limit measure zero). Thus the boundary crossing probability for the asymptotic model R is an approximation of the boundary crossing probability for the "real" situation R_n.

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REFERENCES

[1] P. BALDI, P. BOUGEROL and P. CRÉPEL, *Théorème central limite local sur les déplacements de R^d*, C. R. Acad. Sci. Paris. t. **283**, sér. A, (1976), 53–55.

[2] P. BILLINGSLEY, Convergence of Probability Measures, Willey, New York, 1968.

[3] P. CRÉPEL, *Marches aléatoires sur le groupe des déplacements du plan*, C. R. Acad. Sci. Paris, t. **278**, sér. A, (1974), 961-964.

[4] ———, *Recurrence des marches aléatoires sur les groupes de Lie*, to appear.

[5] ——— and *E. Le Page, Loi du lorarithme itéré fonctionelle pour des marches aléatoires sur le groupe des déplacement de R^d*, C. R. Acad. Sci. Paris, t. **281**, sér. A, (1975), 475–478.

[6] ——— and *A. Raugi, Théorème central limite sur les groups nilpotents*, C. R. Acad. Sci. Paris, t. **281**, sér. A, (1975), 605–608.

[7] L. G. GOROSTIZA, *Some limit theorems in renewal theory*, Bol. Soc. Mat. Mexicana, Vol. **17** (1972), 32–39.

[8] ———, *An invariance principle for a class of* d-*dimensional polygonal random functions*, Trans. Amer. Math. Soc., Vol. **177** (1973), 413–445.

[9] ———, *The central limit theorem for random motions of* d-*dimensional Euclidean space*, Ann. Probab., Vol. **1,** No. 4 (1973), 603–612.

[10] ———, *Convergence of transport processes with radially symmetric direction changes, and chain molecules*, J. Appl. Probability, Vol. **12** (1975), 812–816.

[11] T. LINDVALL, *Weak convergence of probability measures and random functions in the function space* D[0, ∞), J. Appl. Probability, Vol. **10** (1973), 109–121.

[12] N. M. NEWMARK and E. ROSENBLUETH, Fundamentals of Earthquake Engineering, Prentice-Hall, New Jersey, 1971.

[13] G. C. PAPANICOLAU and R. HERSH, *Some limit theorems for stochastic equations and applications*, Indiana U. Math. J., Vol. **21,** No. 9 (1972), 815–840.

[14] O. A. RASCÓN, Modelo estocástico para simular registros de temblores en terreno duro. Tesis, UNAM, México, 1968.

[15] E. ROSENBLUETH and J. I. BUSTAMANTE, *Distribution of structural response to earthquakes*, J. Eng. Mech. Div., Proc. Amer. Soc. Civ. Eng., Vol. **88,** No. EM3, (1962), 75–106.

[16] B. ROYNETTE, *Théoreme central limite pour le groupe des déplacements de R^d*, Ann. Inst. H. Poincaré, Vol. X, No. 4 (1974), 391–398.

[17] ———, *Marches aléatoires sur le groupe des déplacements de R^d*, Z. Wahrscheinlichkeits und Verw. Gebiete, Vol. **31** (1974), 25–33.

[18] V. N. TUTUBALIN, *The central limit theorem for random motions of Euclidean space*, Vestnik Moskov. Univ. Vol. **22,** No. 6 (1967), 100–108 (in Russian). Also in Select. Transl. Math. Statist. Probab., Vol. **12** (1973), 47–57.