A NOTE ON THE LIMITS OF BRANCHING PROCESSES*

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1. Introduction

Little is known about the relation between the limits of the Bellman-Harris process and the embedded Galton-Watson process in the supercritical case. However, some information can be obtained in a simple way. In this note we obtain a functional equation for the joint moment generating function of the two limits, and derive some results concerning their conditional expectations.

2. Results

See [2] for definitions.

For a supercritical Bellman-Harris population, let

Z_t = the population size at time t,

G = the offspring lifetime distribution,

ξ_n = the size of the n-th generation,

$\{p_k\}$ = the offspring production distribution,

$m = \Sigma k p_k$ (>1),

α = the Malthusian parameter ($m \int_0^\infty e^{-\alpha x} dG(x) = 1$),

$c = (m-1)/(\alpha m^2 \int_0^\infty x e^{-\alpha x} dG(x))$.

We will assume that $Z_0 = 1$, $p_0 = 0$, $p_k < 1$ for any k, G is non-lattice and $G(0+) = 0$.

$\{Z_t, t \geq 0\}$ is the *Bellman-Harris process* and $\{\xi_n, n = 1, 2, \cdots\}$ is the *embedded Galton-Watson process*. It is well-known that *if* $\Sigma p_k k \log k < \infty$, *then the limits*

$$\lim_{t\to\infty} c^{-1} e^{-\alpha t} Z_t = W_1 \quad \textit{and} \quad \lim_{n\to\infty} m^{-n}\xi_n = W_2$$

exist a.s., are positive a.s., absolutely continuous and have mean 1. ([2], p. 9, 52, 172; [3], p. 41).

The question is how are W_1 and W_2 related? (Both are defined on the sample space of all family histories).

PROPOSITION. *Let* $\psi_{W_1,W_2}(u_1, u_2) = Ee^{-(u_1 W_1 + u_2 W_2)}$, $u_1, u_2 \geq 0$. *If* $\Sigma p_k k \log k < \infty$, *then* ψ_{W_1,W_2} *satisfies the functional equation*

$$\psi_{W_1,W_2}(u_1, u_2) = \int_0^\infty f[\psi_{W_1,W_2}(u_1 e^{-\alpha x}, u_2 m^{-1})]\, dG(x),$$

where f *is the offspring production generating function* $f(s) = \Sigma p_k s^k$.

Observe that this functional equation contains both of the well-known equations for W_1 and W_2 (by seting $u_1 = 0$ or $u_2 = 0$; [2], p. 10, 172). The equation for W_1 was obtained by Athreya [1] in a different way.

* This work was partially supported by CONACYT grant PNCB-000060

Also, we have $E[W_1 \mid W_2] = W_2$ a.s., whence, if $\Sigma k^2 p_k < \infty$, Cov $(W_1, W_2) = \sigma^2/(m^2 - m)$, where σ^2 is the variance of $\{p_k\}$. (The condition $p_k < 1$ for any k is used only in the proof of this result).

If G is an exponential distribution, we obtain

$$E[W_2 | W_1] = \frac{1}{f_{W_1}(W_1)} \mathscr{L}^{-1}\{(-\psi'_{W_1})^{1/m}\}(W_1) \text{ a.s.},$$

where f_{W_1} is the density of W_1, $\psi_{W_1}(u) = Ee^{-uW_1}$, and $\mathscr{L}^{-1}\{\ \}(x)$ denotes inverse Laplace transform evaluated at x. (This holds also if $p_k = 1$ for some k).

As a simple exercise applying the latter result one can show that $p_k = 1$ $(k > 1)$ if and only if W_1 has the gamma distribution $\Gamma(1/(k-1), 1/(k-1))$, and this is the only way that W_1 can be gamma distributed.

3. Proofs

We use the following additional notation:

$Z_t^{\theta i}$ = the number of descendants at time t of the i-th offspring of element θ,
Θ_n = the set of elements of the n-th generation,
Θ_n^{θ} = the set of n-*th* generation members that descend from element θ,
ξ_n^{θ} = the size of Θ_n^{θ},
$\tau(\theta)$ = the time of death of element θ,
$\mathscr{F}_n$ = the σ-algebra generated by the family tree up the n-*th* generation.

LEMMA. *Let* $X_n = \Sigma_{\theta\in\Theta_n} e^{-\alpha\tau(\theta)}, n = 1, 2, \cdots$. *If* $\Sigma p_k k \log k < \infty$, *then*

$$m \lim_{n\to\infty} X_n = W_1 \text{ a.s.}$$

Proof. For $t > \max_{\theta\in\Theta_n} \tau(\theta)$, $Z_t = \Sigma_{\theta\in\Theta_n} \Sigma_{i=1}^{\xi^{\theta}_{n+1}} Z_t^{\theta i}$,

$$\text{so } c^{-1}e^{-\alpha t}Z_t = \Sigma_{\theta\in\Theta_n} e^{-\alpha\tau(\theta)} \Sigma_{i=1}^{\xi^{\theta}_{n+1}} c^{-1}e^{-\alpha(t-\tau(\theta))}Z_t^{\theta i}.$$

Taking limit as $t \to \infty$ we obtain $W_1 = \Sigma_{\theta\in\Theta_n} e^{-\alpha\tau(\theta)} \Sigma_{i=1}^{\xi^{\theta}_{n+1}} W_1^{\theta i}$ a.s., where the $W_1^{\theta i}$ are independent of $\mathscr{F}_n$ and each other, and distributed as W_1. Hence $E[W_1 | \mathscr{F}_n] = m X_n$ a.s., and therefore, by martingale theory, $m X_n \xrightarrow{a.s.} E[W_1 | \cup_n \mathscr{F}_n] \overset{a.s.}{=} W_1$.

Proof of the Proposition. Clearly, $X_n = e^{-\alpha\beta} \Sigma_{\theta\in\Theta_1} \Sigma_{\theta'\in\Theta_n^{\theta}} e^{-\alpha(\tau(\theta')-\beta)}$, where β is the lifetime of the original parent, and $m^{-n}\xi_n = m^{-1} \Sigma_{\theta\in\Theta_1} m^{-(n-1)}\xi_n^{\theta}$. Taking limits as $n \to \infty$, using the lemma, $W_1 = e^{-\alpha\beta} \Sigma_{\theta\in\Theta_1} W_1^{\theta}$ a.s. and $W_2 = m^{-1} \Sigma_{\theta\in\Theta_1} W_2^{\theta}$ a.s., where the $(W_1^{\theta}, W_2^{\theta})$ are independent and distributed as (W_1, W_2). Therefore $Ee^{-(u_1W_1+u_2W_2)} = Ee^{-\Sigma_{\theta\in\Theta_1}(u_1e^{-\alpha\beta}W_1^{\theta}+u_2m^{-1}W_2^{\theta})}$.
By conditioning on ξ_1 and β the proof is completed.

Now we take partial derivative with respect to the first argument in the functional equation and obtain, for $u_1 = 0$ and $u_2 = u$, $Ee^{-uW_2}W_1 = f'(Ee^{-um^{-1}W_2})Ee^{-um^{-1}W_2}W_1m^{-1}$. On the other hand, $Ee^{-uW_2} = f(Ee^{-um^{-1}W_2})$, so $Ee^{-uW_2}W_2 = f'(Ee^{-um^{-1}W_2})Ee^{-um^{-1}W_2}W_2m^{-1}$. Therefore, denoting $T(u) = Ee^{-uW_2}W_1/Ee^{-uW_2}W_2$, we have $T(u) = T(um^{-1})$. Hence $T(u) = T(um^{-n})$ for all n, and in conclusion $T(u) = T(0) = 1$ for all $u \geq 0$. That is, $Ee^{-uW_2}W_1 = Ee^{-uW_2}W_2$, $u \geq 0$, or $\int_0^\infty e^{-ux}E[W_1 \mid W_2 = x]f_{W_2}(x)\,dx = \int_0^\infty e^{-ux}xf_{W_2}(x)\,dx$, $u \geq 0$, where f_{W_2} is the (strictly positive) density of W_2. It follows from the uniqueness of the Laplace transform that $E[W_1 \mid W_2 = x] = x$ Lebesgue a.s., or $E[W_1 \mid W_2] = W_2$ a.s. Using this, we have $E\,W_1W_2 = E\,W_2E[W_1 \mid W_2] = E\,W_2^2$, so Cov $(W_1, W_2) =$ Var $W_2 = \sigma^2/(m^2 - m)$ ([2], p. 9). (The latter result may be found also by taking mixed derivative in the functional equation. On the other hand, the functional equation does not need to be used; one can start by substituting in $Ee^{-uW_2}W_1$ the expressions for W_1 and W_2 in terms of W_1^θ and W_2^θ given in the proof of the Proposition).

The proof of the expression for $E[W_2 \mid W_1]$ is similar, but a little more elaborate. The idea is that if the functions Φ and Ψ are defined by $\Phi(u) = u^{1/(m-1)}Ee^{-uW_1}W_2$, $\Psi(u) = u^{m/(m-1)}Ee^{-uW_1}W_1$, $u \geq 0$, then one shows that $\Phi(u)/\Psi(u)^{1/m} \equiv 1$, so $Ee^{-uW_1}W_2 = (Ee^{-uW_1}W_1)^{1/m}$, and the result follows.

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